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A New and Comprehensive

SYSTEM

OF

Mathematical Institutions,

Agreeable to the

PRESENT STATE

OF THE

Newtonian Mathesis.

VOL. I.

Containing the Inſtitutes, or Principles of

I. ARITHMETIC, Vulgar and Decimal.
II. LOGARITHMS.
III. ALGEBRA.
IV. GEOMETRY.
V. Of Plain TRIGONOMETRY.
VI. Of CONIC SECTIONS.
VII. FLUXIONS.

By Benjamin Martin.

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MDCCLIX.

G.R III
BRITISH MUSEUM

TO

HIS ROYAL HIGHNESS

GEORGE AUGUSTUS,

PRINCE of WALES.

SIR,

IN every Inſtitution of Government, founded on juſt Principles, the Felicity of the Prince and the People muſt neceſſarily be reciprocal, as it reſults equally from a wiſe and benevolent Exerciſe of Dominion in the one, and good Senſe, and rational Subjection, in the other: This is a Truth verified by a Reflexion on the Fate of Nations in general, but more eſpecially confirmed by that of our own, as well in the laſt as preſent Reign, in which the Bleſſings of Nature have been accumulated on the *Britiſh* Nation almoſt to Profuſion, and we may ſay, with more Juſtice than could be ſaid of them in former Times, that the People would, if poſſible, be too happy were they truly ſenſible of their preſent bliſsful Situation in its utmoſt Extent. But to do ourſelves Juſtice, it muſt be allowed, we are not altogether inſenſible of our Happineſs; for I, with the higheſt Pleaſure, aſſure Your Royal Highneſs, that in more than half the great Towns in *England*, and among all Ranks of People, I have been a conſtant Eye and Ear-Witneſs of their univerſal Joy and Satisfaction with their preſent Condition; of their extraordinary Eſteem and Regard for the Perſon, Title and Government of his preſent Moſt Gracious Majeſty, and their moſt exalted Hopes, and higheſt Confidence in their future Sovereign: This good Diſpoſition, permit me, Great Sir, to ſay, is the natural Conſequence of encouraging the Studies of uſeful Arts and Sciences, Learning and Humanity; for theſe furniſh the Prince with Notions and Principles of Wiſdom, Religion, Virtue, and Liberty, and ſecure the People from the Attacks of Ignorance, Barbarity, Superſtition and Impoſture; and in every Perſon they produce a rational and noble Propenſity towards promoting the general Good of the Community, and the Promulgation of the Sciences among all Ranks and Orders of Men, and inculcate on their Minds Principles that will not fail to render them good Subjects: As this is the profeſſed Deſign of theſe Papers, I humby preſume they will be acceptable to Your Royal Highneſs, and ſhall for ever eſteem it the higheſt Honour that I am permitted to offer them to Your Highneſs's Inſpection. That Heaven may preſerve his preſent Sacred Majeſty to the lateſt deſirable Period of Life, and then Your Highneſs aſcend the *Britiſh* Throne and long reign the happieſt, as well as the greateſt, Monarch of the World, is the inceſſant Prayer of,

SIR,

Your Royal Highneſs's

Moſt dutiful, devoted,

And obedient humble Servant,

B. Martin.

ON THE USEFULNESS OF MATHEMATICAL LITERATURE:

A POEM.

——— ——— ——— ——— ——— ——— Ni
Poſces ante diem librum cum lumine, ſi non
Intendes animum ſtudiis et rebus honeſtis;
Invidiâ vel amore vigil torquebere. ——

HOR. Epiſt. 2. Lib. I.

HAIL! heav'nly Science, *bright*, ſeraphic Truth,
My Soul's Delight, the Labour of my Youth;
O! prove propitious to my artleſs Lays,
Which I devote with Pleaſure to thy Praiſe.
When dread Jehovah *form'd the Worlds above,*
Aſſign'd their Poſts, and bid the Syſtem *move*;
The great, omnipotent, eternal Cauſe,
Commanded thee to regulate their Laws:
(Obedience waits the Word) thy Pow'r to prove,
In perfect Muſic at thy Will they move.
In Afric's *Plains where antient* Nilus' *Flood,*
Draws up the Marks, and ſtains the Lands with Blood;

To teach weak Mortals, Property to ſcan,
Down came Gëometry *and form'd a Plan.*
Mankind, poſſeſs'd of this celeſtial Art,
To diſtant Climes its Influence impart;
Conſtrain the Woods o'er Ocean's *Face to rove,*
And mete the ſpangled Firmament above:
Succeeding Heads, the Properties deviſe
Of Lines on Lines, that meet our wond'ring Eyes;
All Nature ſeem'd ſubmiſſive to the Skill,
And form'd obedient to the human Will.
The Property *laid down, the* Proof *was clear,*
But how to make new Properties *appear*
Was known to few,—(the Joys that Genius brings
Are far ſuperior to all earthly Things;
But deep Invention *ſeldom wills to ſtray*
Amongſt the Proud, the Buſy, or the Gay)
An Art, by which deep Secrets are reveal'd,
Some Antients knew, but carefully conceal'd;
How far by this they clear'd Invention's *Way*
Is hard for modern Geometers *to ſay:*
Its Uſe in Lines in ſome Degree they knew,
As Theon's *Reas'nings evidently ſhew.*
Indeed, what might not Greeks *effect, in Days,*
When Monarchs crown'd Philoſophers with Bays;
When ſolid Learning flouriſh'd thro' the State,
And to be wiſe, was reckon'd truly great:
No foreign Tongue perplex'd their happy Youth,
They taught them Virtue, *and her Siſter* Truth;
Soul-moving Eloquence, ſound moral Rules,
And thou, O Science! *grac'd their public Schools.*
But Sages hold, there's in all mundane Things,
The Cauſe, from which their Diſſolution ſprings;
So Greece *had kept her Learning, and her Fame;*
Had not falſe Sophiſtry *aſſum'd thy Name:*
O'er ſolid Reaſon, rul'd without controul;
And groſs Stupidity hood-wink'd the Soul:
Then, Truth's *bright Eſſence fled the human Race,*
And Learning gone, Barbarity took Place.

But to deſcend thro' dark, unwelcome Times,
To happier Days, in theſe veſperian Climes;
Where Truth's *fair Goddeſs animated Man,*
For Science ſelf we in a Newton *ſcan!*
Amazing Genius! Whoſe prolific Thought,
Nature unveil'd, and deep Fluxonia *taught:*
An Art, that makes the hardeſt, antique Rules,
But boyiſh Problems, in our modern Schools.
Theſe neighb'ring Nations to thee, Science, *owe,*
The new-found Regions, whence their Riches flow;
Who taught Columbus *Ocean's Face to ſweep,*
Or kindred Cabots *how to plow the Deep,*
But thou alone?—And from this Source there ſprings,
The Pow'r, and Grandeur of European *Kings.*
Mankind, by thee, to fartheſt India *roam,*
Whence orient Pearls, Gems, Silks, and Spices come:
Where would Britannia's *Ships, her Commerce be;*
Her Pow'r, her All———O Science, *but for thee?*
The meaneſt Ruſtic craves thy powerful Aid,
To mete the Labour of his toilſome Spade:
Thoſe grand Machines whence Manufactures flow,
(Employ for Thouſands) their Invention owe
To thee:——Theſe Engines Collieries demand,
By Art obſtetric form'd, fair Science plann'd:
Works to force Water, 'Gins to conquer Fire,
Air-mills, and aqueous; all thy Aid require:
Clocks, noting Time's inceſſant circling Courſe,
Balliſtic Engines all-deſtroying Force,
The Orrery ſublime, th' Armillary Sphere,
Glaſſes, that weigh the preſſing, ambient Air,
Sun-dials, optic Inſtruments benign,
Perſpective Drawings, and the Globes are thine.
Except religious Comforts, Naught below
Equals the Bliſs that Science' *Vot'ries know:*
Theſe happy Mortals, Pallas, *by thy Pow'r,*
To diſtant Worlds in Contemplation tour;
What mental Bliſs to view the Order giv'n,
To thoſe bright Orbs; the Ornament of Heav'n!

Not fond Careſſes, nor each charming Grace,
That plays on blooming Celia*'s matchleſs Face;*
Enamel'd Meadows, nor luxuriant Fields,
Give half the Pleaſure, Truth*'s fair Goddeſs yields.*
Winter, *will ſtrip the happieſt ſylvan Shades;*
Old Age, or Sickneſs, ſpring-like Beauty fades;
But thoſe refin'd Enjoyments Science *brings,*
Like 'liv'ning Sol, *are everlaſting Things:*
Compleat the reas'ning Faculties, and ſave
The Soul that Trouble, when beyond the Grave:
  Think! then, ye Mortals, of your murder'd Hours,
In hurrying Cities, or in rural Bow'rs;
Caſt off your Indolence, your Minds careen,
And learn fair Science *from her* MAGAZINE.

CHARLES WILDBORE.

INSTI-

INSTITUTIONS
OF
ARITHMETIC.

The Introduction.

1. MATHESIS is the Science or Doctrine of *Quantity*, whoſe various Relations and Affections it contemplates, and gives Rules for making an *Eſtimate*, or *Computation* thereof; which it expreſſes in different Ways, and by various kinds of Symbols or Characters, which is called *Notation.*

2. *Quantity* is every thing which we can conceive to have any Magnitude or Parts, or, properly ſpeaking, it is any thing concerning which we can aſk the Queſtion, *how much?* Or, *how great?* It is often diſtinguiſhed into *continued* and *diſcrete.*

3. *Continued Quantity* is that whoſe Parts are all contiguous, or adhere together, and make but *one Whole*, as a *Shilling*, a *Stone*, a *Sheep*, &c. this is the Subject of that Part of *Matheſis* which is called *Geometry.*

4. *Diſcrete Quantity* is that whoſe Parts are not continuous, but ſeparate from each other, and make what Logicians call a *collective Idea*, or *Whole*; thus *one Pound* conſiſts of 20 *Shillings*; a *Flock* of many Sheep, *&c.* this is the Subject of the other Part of Matheſis called *Arithmetic.*

5. Notation is of three Kinds, *viz. Numerical Specious*, and *Linear*; *Numerical Notation* is the Repreſentation of Quantity or Quantities by thoſe Characters we call *Numbers* or *Digits*, which are in all Ten, *viz.* 0, Cypher; 1, One; 2, Two; 3, Three; 4, Four; 5, Five; 6, Six; 7, Seven; 8, Eight; 9, Nine. Theſe are uſed in common Arithmetic.

6. *Specious Notation*, is that wherein *Species*, or *Letters* are made uſe of to repreſent Quantities, as the Letters of the Alphabet, *a*, *b*, *c*, or *A*, *B*, *C*, *&c.* as in *Algebra*. Or the ſame Letters with Points over them, thus, $\dot{x}$, $\dot{y}$, $\dot{z}$, in that Part of the Science called *Fluxions*.

7. *Linear Notation*, is the Repreſentation of Quantities by *Lines*, and *Figures* compoſed of *Lines*, as is done in all the Parts of *common Geometry*.

8. The Quantities conſidered in *Arithmetic* are called *Numbers*, of which there are two Sorts, *whole* and *broken*, which are otherwiſe called *Integers* and *Fractions*. The leaſt whole Number is *Unity*, or 1 One; that is, any *one Thing* is called an *Unite*; and *Nothing*, or *Nullity* is repreſented by the Cypher 0.

9. A Number of Units under Ten, is repreſented by a ſingle Digit, as 2, 5, 7, *&c.* but *ten Units* are deſigned by the firſt Digit 1, with a Cypher annexed to the right Hand, thus, 10. And as an Unit is made Ten by one Cypher annexed, ſo it is made *Ten times Ten*, or an *Hundred*, by two annexed Cyphers, *viz.* 100; ſo another Cypher makes *Ten Hundred*, or *One Thouſand*, *viz.* 1000, and ſo on as in the following Table.

1	Unit
10	Ten
100	Hundred
1000	Thouſand
10000	Ten Thouſand
100000	Hundred Thouſand
1000000	Million
10000000	Ten Millions
100000000	Hundred Millions
1000000000	Thouſand Millions.

10. If the Number conſiſts not of even Tens, then ſuch a Digit is annexed to the Unit as will expreſs the ſaid Number; thus Seventeen is expreſſed by 17. Alſo *twice Ten*, or *Twenty*, is expreſſed by the Figure 2 and a Cypher, thus, 20; and *Thirty* by 30; *Forty* by 40; and ſo on to an Hundred. The intermediate Numbers are alſo expreſſed by annexing proper Digits in the Place of the Cypher; thus 25 is Twenty-five; 37 Thirty-ſeven, *&c.*

11. From hence we obtain the Method of *Enumeration*, or expreſſing the Number of Quantities contained in any given *Sum*, as ſhewn below.

NUMERATION TABLE.

Hundred of Millions	Tens of Millions	Millions	Hund. of Thouſ.	Tens of Thouſ.	Thouſands	Hundreds	Tens	Units	
								1	One
							2	1	Twenty-one
						3	2	1	Three Hundred, Twenty-one
					4	3	2	1	4 Thouſand, 3 Hundred, 21
				5	4	3	2	1	54 Thouſand, 3 Hundred, 21
			6	5	4	3	2	1	6 Hundred, 54 Thouſand, 3 Hundred 21
		7	6	5	4	3	2	1	Seven Million, 654 Thouſand, 321
	8	7	6	5	4	3	2	1	87 Million, 654 Thouſand, 321
9	8	7	6	5	4	3	2	1	987 Million, 654 Thouſand, 321.

Here it is plain, that in order to numerate the Figures in any Sum, you have only need to mention firſt each Figure, and then the Place in which it ſtands, according to its Name of Valuation in the Table, in the ſame Manner as you ſee done for each Sum, or Line of Figures in the Table on the Right-hand Side. Thus for Inſtance, the Sum 850943 you read or value thus, 8 Hundred, 50 Thouſand, 9 Hundred, 43; or thus at twice, 850 Thouſand, 943; and ſo the Sum 406528035 is thus read, 406 Million, 528 Thouſand, 35; and ſo of others.

12. As an Unit may have its Value encreaſed ten Times, by annexing a Cypher to the Right-hand, ſo its Value is diminiſhed in a *ten-fold Proportion* by prefixing Cyphers thereto; thus

0,1 is one Tenth
0,01 is one Hundreth } of an Unit.
0,001 is one Thouſandth

Thus alſo { 0,7 is Seven Tenths.
0,53 is Fifty-three Parts of an Hundred.
0,375 is Three Hundred Seventy-five Parts of a Thouſand.

13. In this Caſe, the Cypher on the Left, cut off with a Comma (,) ſtands in the Unit's Place, and ſhews the Number does not amount to Unity, but is a certain Number of ſuch Parts as the Unit contains 10, or 100, or 1000, *&c.* and theſe Parts are

expreſſed by the Figures on the Right-hand of the Comma. This kind of Notation of the Parts of a broken or divided Unit is called *Decimal*, (from *Decem*, Ten) and thoſe Parts of Unity are called *Decimal Numbers* or *Decimal Fractions*.

14. Sometimes a Number conſiſts of *Integers* and *Decimals* together, and is then called a *mix'd Number*; thus 7,3 is Seven and three Tenths; 84,53 is Eighty-four and Fifty-three Parts of an Hundred of another; and ſo of others. That Part of the Science which treats of theſe Numbers is called *Decimal Arithmetic*.

15. If Unity be divided in any other than a *ten-fold Proportion*, then another Species of Computation will enſue; thus in Aſtronomy we divide a Degree into 60 equal Parts or Minutes; theſe Minutes are each divided into 60 *Seconds*; each Second into 60 *Thirds*; and ſo on to *Fourths*, *Fifths*, &c. And they are thus denoted, *viz.* $35^{\circ} : 47' : 31'' : 23'''$; *Thirty-five Degrees*, *Forty-ſeven Minutes*, *Thirty-one Seconds*, *Twenty-three Thirds*. The Rules for managing theſe Numbers is called *Sexagenary* or *Sexageſſimal Arithmetic*.

16. It frequently happens that we are obliged to divide an Unit indefinitely, or into any Number of Parts as Occaſion requires for comparing a Part with the whole Unit, in Parts of ſuch a Diviſion: In this Caſe, the Way to expreſs ſuch a Fraction, is to place the Unit divided into its whole Number of Parts below a Line, and the Parts of the Unit which are given above it; thus $\frac{3}{4}$ is three Parts of ſuch as the Unit contains Four of; and $\frac{5}{13}$ is five Thirteenths of the whole Unit. Theſe are called *Vulgar Fractions*.

17. A Vulgar Fraction is ſaid to be *pure*, when it conſiſts only of fractional Parts, as $\frac{1}{2}$, $\frac{13}{43}$, $\frac{23}{751}$, *&c.* and *mixed*, when joined with Integers, as $5\frac{1}{7}$, $23\frac{4}{9}$, $1\frac{1}{751}$, *&c.*

18. The Number placed below the Line, is called the *Denominator* of the Fraction, becauſe it denominates the Fraction, or Number of Parts into which Unity is broken or divided; and the Number above the Line is called the *Numerator*, becauſe it enumerates or ſhews how many of thoſe Parts make the Fraction propoſed.

19. The Fraction is ſaid to be *proper*, when the Numerator is leſs than the Denominator, as $\frac{3}{4}$; but *improper*, when the Contrary happens; as $\frac{4}{3}$, $\frac{13}{9}$, *&c.*

20. When any two Quantities are compared together, to obſerve the Relation of their *Magnitude*, ſuch Compariſon is called a Ratio; and is thus expreſſed, a : b; of this Ratio, the firſt Term (a) is called the *Antecedent*, and the latter (b), the *Conſequent*.

21 When any two Quantities have the ſame Ratio with any other two, it is denoted by this Character :: thus a : b :: c : d; the Quantity (a) is to (b) as (c) is to (d); which are therefore ſaid to be *analogous* or *proportionate*; and ſuch a Compariſon, or Expreſſion, is call'd *Analogy* or *Proportion*.

22. When any Calculation is to be made, it is done either by *Addition*, *Subſtraction*, *Multiplication* or *Diviſion* of Quantities; which four fundamental Rules are called the *Algorithm* of Quantities, and which we now proceed to explain.

Characters for Abbreviation explained,

viz.	+	ſignifies	More; as $3+4$, is 3 added to 4
	−		Leſs; as $3-4$, is 3 taken from 4
	×		Multiplied by; as 3×4, is 3 multiplied by 4
	÷		Divided by; as $3\div4$, is 3 divided by 4, or $\frac{3}{4}$
	=		Equal to; as $a=4$, is a equal to 4.

CHAPTER I.

ADDITION of INTEGERS, or WHOLE NUMBERS.

23. ADDITION of Numbers conſiſts in adding together all the Units contained in ſeveral particular Numbers, properly diſpoſed, into one *Sum*, *Aggregate*, or *Total*, expreſſing the Value of all together. And this is performed in the following Manner, *viz.*

24. Let the ſeveral particular Sums, or Numbers, be diſpoſed one under another in ſuch a Manner that the Place of Units, Tens, Hundreds, *&c.* in each may conſtitute a perpendicular Column of Figures; thus let it be required to add together the Numbers 57, 762, 5389, 97615; in order to do this they muſt firſt be diſpoſed thus,

$$\begin{array}{r}57\\762\\5389\\97615\\\hline\end{array}$$

25 The Numbers placed, as above, you proceed to add together by the following

RULE,

Reckon up all the Digits in the firſt, or Right-hand, Column, and obſerve, for every Ten to carry One to the Place of Tens in the ſecond Column, ſetting down the remaining Digits under the firſt Column of Units: Thus, $5+9+2+7=23$, which is thus expreſſed, *five* more *nine* more *two* more *ſeven* is *equal* to *twenty-three*, in which Sum there are *two Tens* and 3 over; you muſt then ſet down the 3 and carry Two to the next Place of Tens, and proceed as before; thus, $2+1+8+6+5=22$; here again are *two Tens*, and Two over to be ſet down under the ſecond Column; then carrying the Two to the third Column of Hundreds, you ſay again, $2+6+3+7=18$; here is but *one Ten*, and 8 to be ſet down; then carrying One to the next Column, ſay $1+7+5=13$; here again is *one Ten*, and 3 to be ſet down; laſtly, carry One to the laſt Place, and ſay $1+9=10$, which Number, be it what it will, is always ſet down, and the Sum total is compleat in one Number, as in the Examples.

57	6475
762	9830
5389	2764
97615	5937
103823	25006

26. The Reaſon why you carry *Ten* from every Column to the next, is becauſe the Value of the Figures in each Column encreaſes in a ten-fold Proportion, as is evident from Article 9; and the Digits ſet down under each Column are *Units*, *Tens*, *Hundreds*, *Thouſands*, *&c.* according to Inſt. 11. which will expreſs the Value of all the Columns ſeverally collected and added together. Thus in the firſt Example, the Sums of each Column will ſtand thus, *viz.*

23	Column of Tens
200	——— of Hundreds
1600	——— of Thouſands
12000	——— of Ten Thouſands
90000	——— of Hundred Thouſands.
103823	= Total Value.

27. More Examples of Addition are the following:

5729	5009	847593
43605	120	215475
9834	369	176843
30546	1298	467890
75102	57306	150362
164816	64102	1858163

28. In the Addition of ſeveral Sums of *Money*, *Meaſure*, *Weight*, *&c.* you obſerve one general Rule, *viz.* *to collect all the Units of one Species together; and then for every Number of thoſe Units, which make one of the next Species, you add One to the next Column, ſetting down the Remainder, as above.* In Money the Species are thus denoted,

viz.	*f.*	for	Farthings	of which	4=1 Penny.
	d.		Pence		12=1 Shilling.
	s.		Shillings		20=1 Pound.
	L.		Pounds		

Thus in the following Example, for every 4 Farthings you carry one Penny to the Column of Pence; for every 12 Pence you carry One to the Column of Shillings, and from thence, for every Twenty, you carry one to the Place of Pounds; which are added as the Integers in Inſt. 27.

Example.

	l.	*s.*	*d.*	*f.*
	37	12	9	3
	125	7	10	2
	750	16	7	1
Total £.	913	17	3	2

29. I ſhall here ſubjoin ſeveral Sums to be added for the Learner's Practice.

l.	*s.*	*d.*	*f.*	*l.*	*s.*	*d.*	*f.*
175	19	6	2	3758	17	11	2
386	18	10	3	279	16	10	3
57	19	4	1	46	8	9	1
6	12	5	2	8	19	7	2

l.	s.	d.	f.	l.	s.	d.	f.
10	16	10	2	1000	16	10	2
175	18	9	0	100	10	11	3
37	14	11	2	90	2	3	2
956	16	10	1	10	12	10	1
47	19	9	3	9	17	5	1
				0	17	4	0

30. Of *Weights* we have ſeveral Sorts, *viz.*

I. TROY WEIGHT.

In which
24 Grains *(Grs.)* make 1 Penny Weight.
20 Penny Wts. *(Pwt.)* 1 Ounce.
12 Ounces *(oz.)* —— 1 Pound. *(lb.)*

31. II. APOTHECARIES WEIGHT.

In which
20 Grains *(Grs.)* make 1 Scruple.
3 Scruples (℈) —— 1 Dram.
8 Drams (ʒ) —— 1 Ounce.
12 Ounces (℥) — — 1 Pound *(lb.)*

32. III. AVOIRDUPOIS WEIGHT.

In which
16 Drams *(Drs.)* make 1 Ounce.
16 Ounces *(Oz.)* —— 1 Pound.
28 Pounds *(lb.)* — — 1 Quarter of an Hundred.
4 Quarters *(Qrs.)* — 1 Hundred Weight.
20 Hundred *(C.)* — 1 Ton.

33. In each of theſe Species the Learner may exerciſe himſelf by the following Examples, *viz.*

In Troy Weight.

lb.	Oz.	Pwt.	Grs.
21	11	17	21
20	9	15	7
8	4	3	23
19	10	19	18

In Apothecaries Weight.

lb.	℥.	ʒ.	℈.	*Grs.*
18	10	7	2	16
9	11	5	0	10
16	9	4	1	10
3	5	6	2	15
25	9	0	1	6

In Avoirdupois Weight.

T.	*C.*	*Qrs.*	*lb.*	*Oz.*	*Drs.*
175	17	3	25	14	13
90	10	2	17	10	12
105	18	1	11	9	4
19	19	0	27	15	3
195	17	1	14	9	10
74	10	3	8	12	9

In Long Meaſure.

34. Our MEASURES are of divers Sorts; the Meaſures of Length are of the following ſpecies.

3 Barley Corns make 1 Inch.
12 Inches — — — 1 Foot.
3 Feet — — — 1 Yard.
5½ Yards — — 1 Rod.
40 Rods — — — 1 Furlong.
8 Furlongs — — 1 Mile.

35. Our CORN MEASURES are the following.

2 Gallons make 1 Peck.
4 Pecks — 1 Buſhel.
8 Buſhels — 1 Quarter.
5 Quarters — 1 Load.

36. Our LIQUID MEASURES are of the following Species, *viz.*

8 Pints make 1 Gallon.
9 Gallons — 1 Firkin of Beer.
8 Gallons — Ditto of Ale.
2 Firkins — 1 Kilderkin.
2 Kilderkins 1 Barrel.
2 Barrels — 1 Hogſhead.

37. But our WINE MEASURE is as follows, *viz.*

8 Pints make 1 Gallon.
63 Gallons — 1 Hogſhead.
2 Hogſheads 1 Pipe or Butt.
2 Butts — 1 Tun.

38. The true original Standard for Meaſures of Capacity or Solidity, is the CUBIC INCH.

Thus	282	Cubic Inches make	1 Ale Gallon.
	231		1 Wine Gallon.
	268¾		1 Corn Gallon.

Alſo 1728 Cubic Inches make one Foot Solid.

39. Our Meaſures of TIME are of the following Denominations, *viz.*

60 Seconds make 1 Minute.
60 Minutes — 1 Hour.
24 Hours —— 1 Day.
7 Days —— 1 Week.
4 Weeks —— 1 Month.
13 Months —— 1 Year.

---

40. A CIRCLE, and alſo the MOTION performed in a Circle, (eſpecially in reſpect of the ECLIPTIC in aſtronomical Affairs) is divided in the following Manner, *viz.*

60 Seconds make 1 Minute.
60 Minutes — 1 Degree.
30 Degrees — 1 Sign.
12 Signs, or
360 Degrees } — The Circle.

In very nice Matters we ſub-divide a Second into 60 Thirds; a Third into 60 Fourths, and ſo on; as mentioned, Inſt. 5.

41. I ſhall ſubjoin the following Example for Practice.

Long Meaſure.

M.	*F.*	*Rds.*	*Yds.*	*F.*
175	5	36	3	2
19	3	27	5	1
76	7	39	4	0
121	0	20	2	1
54	6	17	4	0

M.	*F.*	*Rds.*	*Yds.*	*F.*	*in.*
195	3	27	3	1	10
79	7	30	4	0	11
9	6	29	5	2	9
17	5	20	4	0	7
59	4	18	0	1	6

Time.

Y.	*M.*	*W.*	*D.*	*H.*
175	9	3	2	17
140	12	1	5	21
35	7	0	6	19
9	5	2	4	11
110	10	1	3	8

Y.	*M.*	*W.*	*D.*	*H.*	′	″
143	10	2	5	19	50	37
74	9	3	6	20	7	57
3	4	0	3	1	39	10
19	3	1	2	5	25	41
158	12	2	0	22	30	8

Motion.

Rev.	*Sg.*	*Dr.*	'	"	*Rev.*	*Sg.*	*Dr.*	'	"
175	10	28	51	42	237	11	22	50	40
74	11	17	41	37	41	10	20	9	30
50	9	21	30	58	17	9	17	17	27
9	8	11	35	53	573	5	9	27	35
5	10	5	10	9	29	8	10	15	42
3	5	13	7	16	17	3	8	16	51

42. The beſt Way to know if the Sum be caſt up right, is to do it twice over, beginning at the Bottom, and reckoning upwards the firſt Time; and then beginning at the Top, and reckoning downwards the ſecond Time; and if the *Sum Total* in both Caſes be the ſame, the Work is right, otherwiſe not.

CHAP. II.

SUBTRACTION of INTEGERS.

43. SUBTRACTION is the ſecond Operation in the Art of *Computation*; it conſiſts in finding the Difference between two Numbers, *by taking the Leſſer from the Greater*; and as this is the Reverſe of Addition, ſo the Rule of performing it, is the Contrary of that, *viz.*

RULE.

44. *From each Figure in the upper Line (beginning at the Right Hand) take the Figure in the under Line, if it be leſs, and ſet down the Difference, but if the Figure or Number in the lower Line exceeds that in the upper One which ſtands over it, encreaſe that upper Figure by adding* 10, *if it be in the Place of Integers; but if it be in any Species of Money, Weight, Meaſure,* &c. *you add to it ſuch a Number as makes One of the next Denomination, and then ſubtract as before, and ſet down the Difference, and remember, that every Time you thus encreaſe the upper Figure, you carry* one Unit *to the next Figure in the lower Line.*

45. This Rule will be beſt illuſtrated by the following Examples.

From	57238765	695278675
Take	13106243	434865740
Remains	44132522	260412935

5729438640	109765432016
178089305	16741320785

Money.

l.	*s.*	*d.*
175	17	9
94	13	6
81	04	3

l.	*s.*	*d.*	*f.*
17653	05	11	3
589	17	10	2
17063	08	01	1

l.	*s.*	*d.*	*f.*
9175	17	04	2
1049	19	10	3

l.	*s.*	*d.*	*f.*
3729	17	10	0
3018	17	11	3

Troy Weight.

l.	*Oz.*	*Pwt.*	*Grs.*
131	10	18	14
79	11	19	20

l.	*Oz.*	*Pwt.*	*Grs.*
173	05	00	01
9	10	00	23

Avoirdupois Weight.

T.	*C.*	*Qrs.*	*lb.*	*Oz.*
35	18	2	24	0
34	19	3	25	9

T.	*C.*	*Qrs.*	*lb.*	*Oz.*	*Drs.*
13	00	0	01	00	15
10	17	2	7	09	15

Long

Long Meaſure.

M.	*F.*	*Rds.*	*Y.*	*F.*
173	7	37	4	1
19	7	39	5	0

M.	*F.*	*Rds.*	*Yds.*	*F.*	*In.*
517	6	00	1	2	10
9	0	00	4	2	11

Motion.

Rev.	*Sg.*	*Dr.*	'
15	07	20	40
14	11	29	50

Rev.	*Sg.*	*Dr.*	'	"
10	9	29	59	00
8	10	29	00	59

Time.

Y.	*M.*	*W.*	*D.*	*H.*
157	10	0	6	21
25	12	2	1	23

Y.	*M.*	*W.*	*D.*	*H.*	'	"
125	10	0	0	00	00	01
124	11	3	6	23	59	09

46. Theſe Examples well underſtood, all others of every Sort will be eaſy; and the Reader needs not be told, that the Difference added to the lower Line, or leſſer Number, ought to make the Sum in the upper Line, and will, when the Work is right.

CHAP. III.

MULTIPLICATION of INTEGERS.

47. MULTIPLICATION is the *third Operation* in Arithmetic, and is nothing more than a compendious Way of adding any Number or Sum, any Number of Times to its ſelf. Thus if the Number 135 be added 3 Times to its ſelf, the Sum will be the ſame as the Product of 135 multiplied by 3, as it evident below, *viz.*

Add

Add	135 135 135	Multiply	135
		By	3
Sum	405	Product	405

48. If the ſame Number 135 were to be made 13 Times as great, then you make it firſt, 3 Times as great, *viz.* 405; and after that, 10 Times as great (which is done by annexing a Cypher to the right Hand, Inſt. 9.) *viz.* 1350, and then the Sum of both theſe is that required, *viz.*

The Factors { 135 The Multiplicand.
13 The Multiplier.

1ſt Sum 405
2d Sum 1350

Total 1755 Product.

49. But that a Perſon may be ready at inveſtigating theſe particular Sums, it is abſolutely neceſſary to learn the following Table by heart.

The Multiplication Table.

3 Times { 3 is 9
4 12
5 15
6 18
7 21
8 24
9 27

4 Times { 4 is 16
5 20
6 24
7 28
8 32
9 36

5 Times { 5 is 25
6 30
7 35
8 40
9 45

6 Times { 6 is 36
7 42
8 48
9 54

7 Times { 7 is 49
8 56
9 63

8 Times { 8 is 64
9 72

9 Times 9 81

50. By this Table we find the Sum which ariſes by adding any Digit any Number of Times (under 10) to its ſelf; thus 8 added 7 Times to its ſelf makes 56, and therefore we ſay, 7 Times 8 is 56; and ſo of others. The Learner will take Notice, that 7 Times 5 is the ſame as 5 Times 7, and therefore, it was not neceſſary to make the Table any larger.

51. Having this Table perfectly in Mind, the Operation will be eaſy by the following

RULE.

With the firſt Figure to the Right Hand, in the Multiplier, begin to multiply each Figure in the Multiplicand, proceeding from the Right to the Left; in the Product of each Figure, ſet down the Digits under 10, or between the Tens; and for every 10 contained in ſuch Product, carry One to the next Place. In like Manner proceed with every Figure of the Multiplier; and ſet down the ſeveral Products, one under another, in ſuch a Manner that they may ſtand one Figure backwards, or the firſt Figure of each Line under the ſecond Figure of that above; then add all into one Sum, which will be the Product, as in the following

Examples.

Mult.	1735	5738	4897
by	4	16	384
	6940	34428	19588
		5738	39176
			14691
		91808	
			1880448

52. The Reaſon of the laſt Part of the Rule, *viz.* ſetting the ſeveral Products one Figure back will eaſily appear from Inſt. 48. Becauſe the Cyphers annexed to each Product are here rejected as making no Alteration in the Value of the Product, when thus reckoned up. Thus, in the laſt Example, the Multiplicand 4897, being multiplied by 4, 80, 300, ſucceſſively, makes the Products, 19588, 391760, 1469100, which, added together, make the ſame Number as before; as is evident below

```
   4897
    384
-------
  19588
 391760
1469100
-------
1880448
-------
```

Here the three Cyphers being rejected, as ſuperfluous, the Reſt ſtands as before.

53. If a Cypher be found in the Multiplier, you place a Cypher under it, or annex it to the firſt Figure of the next Product; or you ſet the next Figure two Places back inſtead of one. Thus 1795 multiplied by 307 will ſtand

thus;

```
  1795
   307
 -----
 12565
 0000
5385
------
551065
------
```

or thus;

```
  1795
   307
 -----
 12565
53850
------
551065
------
```

or thus;

```
  1795
   307
 -----
 12565
5385
------
551065
------
```

The laſt Way is moſt compendious, but the 2d is eaſieſt for a Learner, and is generally uſed; the Reaſon of both appears from the firſt Poſition of the Product; for it is evident the 4 Figures 1795 multiplied by 0, gives 4 Places of Cyphers, which are of no uſe to ſet down.

54. I ſhall here ſubjoin a Variety of Examples for the Learner's Obſervation and Exerciſe, both wrought and unwrought, as follow:

```
   1750        7580        80507
     76         520         3050
 ------     -------    ---------
  10500      151600      4025350
 12250       37900     2415210
 ------     -------    ---------
 133000     3941600    245546350
 ------     -------    ---------
```

756940	705059
10090	1000
68124600	705059000
75694000	
7637524600	
579603	1001000
10001	1010
579603	10010000
579603000	10010000
5796609603	1011010000
89704500	900203005
605730	90107004
57986000	100100010
18000500	10010010

55. The vulgar Method of proving the Truth of the Work, s by adding together the Digits in the Multiplicand and Multilier, and caſting out every *Nine*, as often as it occurs, noting e Remainders; then multiplying the Remainders, and caſtng the Nines out of the Product, you note the Remainder in at Caſe: Laſtly, the Nines are caſt out of the Sum of the Diits in the Product, and if the Remainder, in this Caſe, be the me with the laſt mentioned, it is a general Proof that the ork is true. Thus in the firſt Example of Inſt. 54, the two ſt Remainders are 4 and 4; the Remainder in their Product is , which is alſo the Remainder in the whole Product of the Facrs, and ſhews the Work is right. But the genuine Proof of ultiplication is by *Diviſion*.

56. *Compound Multiplication* is ſometimes of Uſe. This is hen the *Multiplicand* conſiſts of differents Parts or Species of

Meaſure, *Weight*, *Time*, *Motion*, &c. As in the Examples following.

	l.	*s.*	*d.*	*lb.*	*oz.*	*pwts.*	*gr.*	*yds.*	*f.*	*in.*
Mult.	15	7	$6\frac{1}{2}$	8	11	16	10	40	2	$10\frac{1}{4}$
By			5				8			12
	76	17	$8\frac{1}{2}$							

But when the Multiplier is large, theſe Operations are much eaſier by *Decimal Arithmetic*, as will be ſhewn farther on.

CHAP. IV.

DIVISION of INTEGERS.

57. BY this Operation we find how often the ſmalleſt of any two given Numbers is contained in the greater, which is therefore but the Reverſe of Multiplication; as for Example, Let it be required to find how often 3 is contained in 405. Then firſt, 3 is contained in 405, 100 Times, with a Remainder of 105; in this Remainder it is contained 30 Times with another Remainder of 15, in which it is contained 5 Times; therefore in all the Parts of 405, the Number 3 is contained, 100+30+5= 135 Times, as in the Operation below. (See Inſt. 47)

Diviſor.	*Dividend.*	*Quotient.*
3)	405	(100
	300	
3)	105	(30
	90	
3)	15	(5
	15	
	..	135

58. Hence you ſee the Reaſon of the compendious Form in the common Operation of this Part of Arithmetic; which therefore is performed by the following

RULE.

RULE.

If the Diviſor be a leſs Number than ſo many Figures taken on the Left Hand in the Dividend make, ſee how often the Former is contained in the latter, and the Figure which expreſſes it, is the firſt of the Quotient; then multiply the Diviſor by the Quotient Figure, and placing the Product under the ſaid Figure or Figures of the Dividend, ſubtract it therefrom; and to the Remainder, annex the following Figure of the Dividend, which divide as before; and thus proceed 'till the whole Dividend be exhauſted; as the Examples following.

```
3 ) 405 ( 135          8 ) 9769 ( 1221
    3 . .                  8
    —                      —
    10                     17
     9                     16
    —                      —
    15                      16
    15                      16
    —                       —
    . .                     . . 9
                                8
                                —
                                1

12 ) 1728 ( 144        135 ) 1755 ( 13
     12                      135
     —                       —
     52                      405
     48                      405
     —                       —
      48                     . . .
      48
      —
      . .
```

59. If it happen that the Diviſor be a greater Number than ſo many of the firſt Figures of the Dividend make, then you take a Number of Places in the Dividend greater by one, and proceed as before, as in the following Examples.

5) 3790 (758
35
—
29
25
—
40
40
—
. .

25) 127900 (5116
125
—
29
25
—
40
25
—
150
150
—
. . .

60. If in any Caſe, the Remainder be ſo ſmall, that when the Figure of the Dividend joined with it, make a Sum leſs than the Diviſor, than a Cypher is to be placed in the Quotient, and another Figure taken down; and then the Diviſion renewed, proceeds as above. Thus

9) 1863 (207
18
—
. . 63
63
—
. .

12) 250836 ($2\dot{0}903$
24
—
108
108
—
. . . 36
36
—
. .

61. If the Diviſor be not contained a whole Number of Times in the Dividend preciſely, there will be a Remainder, when the Diviſion is finiſhed, which is to be placed Fraction-wiſe in the Quotient, together with the Diviſor, as the Denominator, as in the following Caſes.

7) 673 (96$\frac{1}{7}$
63
—
43
42
—
1

13) 12976 (998$\frac{2}{13}$
117
—
127
117
—
106
104
—
2

71)

71) 29754 ($419\frac{5}{71}$
284
135
71
644
639
··5

131 (135076 ($1031\frac{15}{131}$
131
··407
393
146
131
15

62. When the Diviſor has one or more Cyphers at the Beginning, they may be omitted in the Work, provided you ſtrike off as many Figures in the Beginning of the Dividend, which are to be annexed to the Remainder to compleat it; as will appear in the enſuing Operations.

At Length.

70) 5689 ($81\frac{19}{70}$
560
89
70
19

8300) 9850716 (1186
8300
15507
8300
72071
66400
56716
49800
·6916

Contracted.

7|0) 568|9.($81\frac{19}{70}$
56
··8
7
19

83|00) 98507|16 (1186
83
155
83
720
664
567
498
6916

63. Theſe are all the Varieties the Learner will meet with in this Operation; and the following Examples will exerciſe him therein.

175) 89463 (9850) 82768 (
5900) 300012 (15700) 9875830 (
1000) 980005 (60800) 908437200 (

64. To prove the Truth of the Work, multiply the Quotient by the Diviſor, and if the Product be the ſame with the Dividend, the Work is right, otherwiſe not. Note, if there be any Remainder in the Diviſion, it muſt be added to the ſaid Product to make the Proof. Thus in the laſt Example of (61) we have $1031 \times 131 = 135061$, and then the Remainder 15 being added, the Whole makes 135076, the ſame with the Dividend.

65. And, *vice verſâ*, the only true Proof of Multiplication is by Diviſion, as will eaſily appear to any one who underſtands the preceding Operations. Thus in the firſt Example of (54) if you divide the Product 133000 by the Multiplier 76, the Quotient will be the Multiplicand 1750, which proves the Work true.

CHAP. V.

Of VULGAR FRACTIONS.

66. TO fit Vulgar Fractions for Operation, they ſhould be firſt of all reduced to their loweſt Denomination, which is done by finding a common Diviſor by this

RULE.

Divide the Denominator by the Numerator, and the Numerator by the Remainder (if any) and that firſt Remainder by the Second, the ſecond by the Third, and ſo on, till the Remainder be nothing; as in the Inſtances following, viz. $\frac{15}{81}$, $\frac{31}{157}$.

15)

15) 81 (5
75
—
6) 15 (2
12
—
3) 6 (2
6
—
.

31) 157 (5
155
—
2) 31 (15
30
—
1) 2 (2
2
—
.

67. The laſt Diviſor is the *common Diviſor* ſought, by which, if the Fraction be divided, it will be reduced to its loweſt Terms; thus if the Fraction $\frac{15}{81}$ be divided in each Part by 3, it is reduced to $\frac{5}{27}$, which is equal in Value to the former. And becauſe, in the ſecond Example, the Remainder is Unity, it ſhews that Fraction is already in its loweſt Denomination.

68. *Mix'd Numbers* muſt be reduced to a fractional Form, which is done by this

RULE.

Multiply the integral Part by the Denominator of the fractional Part, to which add the Numerator, and the Sum will be a new Numerator; under which write the Denominator, and it makes an improper Fraction of the ſame Value. As in theſe Examples, $5\frac{2}{7}$, $1\frac{3}{11}$, $28\frac{1}{13}$. *Thus* $5\times7=35$, *and* $35+2=37$, *then* $\frac{37}{7}=5\frac{2}{7}$; *alſo* $1\times11=11$, *and* $11+3=14$, *then* $\frac{14}{11}=1\frac{3}{11}$; *laſtly,* $28\times13=364$, *and* $364+1=365$, *then* $\frac{365}{13}=28\frac{1}{13}$. *The Reaſon of all which appears from common Diviſion* (61.)

69. Two or more Fractions are reducible to *one common Denomination*, that is, ſhall have all the ſame Denominator, retaining ſtill their firſt Values, by this general

RULE.

Multiply the Numerator of each Fraction into all the Denominators, but its own, for a new Numerator, and multiplying all the Denominators together, the Product ſhall be a common Denominator to all the Numerators before found.

Thus let it be required to reduce $\frac{2}{5}$, $\frac{7}{13}$, $\frac{17}{73}$, to one common Denomination, the Work will ſtand as below.

2	7	17	73
13	5	13	13
26	35	51	219
73	73	17	73
78	105	221	949
182	245	5	5
1898	2555	1105	4745

Then the Fractions become $\frac{1898}{4745}$, $\frac{2555}{4745}$, $\frac{1105}{4745}$, all of the ſame Value as before.

70. This previous Reduction of Fractions to a common Denominator, renders them eaſy to be added, or ſubtracted; for this is done among the Numerators only of the Fractions thus reduced; for Example, to add the two Fractions $\frac{2}{5}$ and $\frac{7}{13}$ together; theſe reduced (*per* 69) are $\frac{1898}{4745}$ and $\frac{2555}{4745}$, whence $1898+2555=4453$; conſequently $\frac{4453}{4745}=\frac{2}{5}+\frac{7}{13}$, the Sum required. Thus alſo, $\frac{2}{5}+\frac{7}{13}+\frac{17}{73}=\frac{5558}{4745}=1\frac{813}{4745}$. So any other Fractions are added together.

71. In like Manner you ſubtract one Fraction from another; thus $\frac{7}{13}-\frac{2}{5}=\frac{657}{4745}$, becauſe $2555-1898=657$. So alſo $\frac{2}{5}-\frac{17}{73}=\frac{1898-1105}{4745}=\frac{793}{4745}$. And the like of others. When Fractions are thus added or ſubtracted, the fractional Sum or Difference is to be reduced to its loweſt Denominator by the Rule of (66.)

72. To *multiply Fractions* together, is no more than to multiply the Numerators and the Denominators among themſelves, and the Product thence ariſing will be that which is required. Thus $\frac{2}{5}\times\frac{7}{13}=\frac{14}{65}$. Thus alſo $\frac{7}{13}\times\frac{17}{73}=\frac{119}{949}$; ſo $\frac{1}{9}\times\frac{7}{5}=\frac{7}{45}$.

73. To *divide one Fraction* by another, you multiply the Denominator of the Diviſor by the Numerator of the Dividend, for the Numerator of the Quotient; then you multiply the Numerator of the Diviſor by the Denominator of the Dividend for the Denominator of the Quotient. Thus $\frac{14}{65}$ divided by $\frac{2}{5}$ will ſtand thus, $\frac{2}{5})\ \frac{14}{65}\ (=\frac{70}{130}=\frac{7}{13}$; and thus $\frac{7}{13})\ \frac{119}{949}\ (\frac{1547}{6643}=\frac{17}{73}$, by Reduction. Laſtly, $\frac{1}{9})\ \frac{7}{45}\ (\frac{63}{45}=\frac{7}{5}=1\frac{2}{5}$. Theſe Caſes of Diviſion being only the Reverſe of thoſe of Multiplication (72.)

47. Any whole Number is expreſſed Fraction-wiſe, by only writing Unity under it, thus 5 is $\frac{5}{1}$, 17 is $\frac{17}{1}$, &c. And ſo Integers and Fractions may be added, ſubtracted, multiplied or divided in the ſame Manner, as pure Fractions themſelves by the preceding Rules. Thus $\frac{2}{15}\times 5=\frac{2}{15}\times\frac{5}{1}=\frac{10}{15}=\frac{2}{3}$. On the contrary, $5)\frac{2}{3}(=\frac{5}{1})\frac{2}{3}(=\frac{2}{15}$.

CHAP. VI.

The Reduction of Quantities of divers Denominations into One.

75. THE Rule for doing this is to multiply each ſuperior Species by the Number which it contains of the propoſed Species below it. Thus one Pound is reduced to Numbers expreſſing an equal Value in all the inferior Species; as alſo other Quantities, as in the Examples below.

1 *l. Sterling.*
20

20 *Shillings.*
12

240 *Pence.*
4

960 *Farthings.*

1 *lb. Troy.*
12

12 *Ounces.*
8

96 *Drams.*
3

288 *Scruples.*
20

5760 *Grains.*

1 *lb. Avoirdupois.*
16

16 *Ounces.*
16

256 *Drams.*

1 *Tun.*
20

20 *Hundreds.*
4

80 *Quarters.*
28

2240 *Pounds.*
16

35840 *Ounces.*

$$\begin{array}{rl} 1 & \textit{lb. Apothecaries Wt.} \\ \underline{12} & \\ 12 & \textit{Ounces.} \\ \underline{20} & \\ 240 & \textit{Penny Wt.} \\ \underline{24} & \\ 5760 & \textit{Grains.} \end{array}$$

$$\begin{array}{rl} 1 & \textit{Mile} \\ \underline{8} & \\ 8 & \textit{Furlongs.} \\ \underline{40} & \\ 320 & \textit{Rods.} \\ \underline{16\frac{1}{2}} & \\ 5280 & \textit{Feet.} \\ \underline{12} & \\ 63360 & \textit{Inches.} \end{array}$$

$$\begin{array}{rl} 1 & \textit{Day.} \\ \underline{24} & \\ 24 & \textit{Hours.} \\ \underline{60} & \\ 1440 & \textit{Minutes.} \\ \underline{60} & \\ 86400 & \textit{Seconds.} \end{array}$$

$$\begin{array}{rl} 1 & \textit{Circle.} \\ \underline{12} & \\ 12 & \textit{Signs.} \\ \underline{30} & \\ 360 & \textit{Degrees.} \\ \underline{60} & \\ 21600 & \textit{Minutes.} \\ \underline{60} & \\ 1296000 & \textit{Seconds.} \end{array}$$

76. If the Quantity, to be reduced, conſiſts of different Species, the Number of each ſeveral Species is to be taken in, or added to the Product which is of the ſame Kind or Denomination, as you proceed in the Reduction; according to what you ſee here exemplified.

$$\begin{array}{rrrr} \textit{l.} & \textit{s.} & \textit{d.} & \textit{f.} \\ 15 & 17 & 4 & 3 \\ \underline{20} & & & \\ 317 & \textit{s.} & & \\ \underline{12} & & & \\ 3808 & \textit{p.} & & \\ \underline{4} & & & \\ 15235 & \textit{f.} & & \end{array}$$

$$\begin{array}{rrrr} \textit{lb.} & \textit{oz.} & \textit{pwts.} & \textit{grs.} \\ 9 & 7 & 15 & 21 \\ \underline{12} & & & \\ 115 & & & \\ \underline{20} & & & \\ 2315 & & & \\ \underline{24} & & & \\ 9261 & & & \\ \underline{4632} & & & \\ 55581 & & & \end{array}$$

Hence ſeveral Queſtions may be propoſed for the Learner's Exerciſe, as follow.

lb. *oz.* *drs.*
In 35 13 15 Avoirdupoiſe Weight, how many Drams?
T. *C.* *Qrs.* *lb.*
In 17 13 3 25, how many Pounds?
M. *F.* *R.* *F.*
In 57 7 37 15, how many Feet?
D. *H.* ′ ″
In 275 17 39 48, how many Seconds?
R. *S.* ° ′ ″
In 15 10 27 49 53, how many Seconds?
Lds. *Qrs.* *B.* *G.*
In 35 4 7 5, how many Gallons?
Hgs. *Gal.* *Pts.*
In 13 51 7, how many Pints?

77. When we have any large Number expreſſing any Quantity in its lower Species, the ſame may be reduced to any or all the higher Species by Diviſion, being only the Reverſe of the other Proceſs by Multiplication ( ).

Thus ſuppoſe it were required to know how many Pounds, Shillings, Pence and Farthings were contained in 15235 Farthings? 'Tis evident, *if we divide firſt by* 4, *the Quotient will be Pence*; *this divided by* 12, *will quote Shillings*; *and theſe again divided by* 20, *will ſhew the Number of Pounds*; *the Remainders in each Diviſion being the odd Shillings, Pence, and Farthings.* The Work placed after the uſual Manner will ſtand as below.

```
          12     20
4 ) 15235 ( 3808 ( 317 ( 15
    12      36     20
    —       —      —
     32     20    117
     32     12    100
     —      —     ——
   ··35     88     17
     32     84
     —      —
     ·3     ·4
```

Anſwer 15*l.* 17*s.* 4*d.* 3*f.*

Again, In 55581 Grains, how many Pounds, Ounces, Pennyweights, and Grains?

```
             20       12
24 ) 55581 ( 2315 ( 115 ( 9
     48      20       108
     —       —        ———
      75     31         7
      72     20
      —      —
       38    115
       24    100
       —     ———                   lb. oz. pwt. grs.
      141    15            Anſwer   9   7   15   21
      120
      ———
       21
```

After the ſame Manner the Learner may operate the following Queſtions.

In 59786 Grains, how many Pounds, Ounces, Drams, Scruples, and Grains? Troy.

In 197568 Ounces, Avoirdupois, how many Tuns?

In 1000000 Feet, how many Miles, *&c.*

In 2974600″, how many Days?

In 5973864 Pints, how many Load?

CHAP. VII.

Of DECIMAL ARITHMETIC.

78. THE Nature and Notation of Decimal Numbers having been already declared (12, 13, 14,) I ſhall proceed immediately to give the Rules for their Operation. The firſt of which is

ADDITION.

The *Addition* of Decimal Numbers does no way differ from that of Integers, due Care being taken to place all the particular Sums, ſo that the firſt Places of the integral or decimal Parts be exactly under each other, as is ſeen in the following Examples. (See 25, 27.)

5,7	64,75	5,729
76,2	98,30	43,605
538,9	27,64	9,834
9761,5	59,37	30,546
10382,3	250,06	75,102
		164,816

79. If any Number be purely decimal, or has no integral Part, 'tis uſual to put a Cypher in Units Place of Integers, as thus,

0,057	0,5729
0,762	4,3605
5,389	0,9834
97,615	3,0546
103,823	7,5102
	16,4816

80. Cyphers on the Right Hand of a decimal Number avail nothing, and are therefore more elegantly omitted.

Thus inſtead of	0,0100	we write	0,01
	53,9270		53,927
	4,1009		4,1009
	975,2300		975,23
	87,1200		87,12
	1120,3879		1120,3879

SUBTRACTION.

81. The ſame Precautions being obſerved, *Subtraction* of Decimals is performed in all Reſpects like that of Integers, as in the Examples following. See (44.)

5723,8765	69527,8675
1310,6243	43486,574
4413,2522	26041,2935

572,0987

572,0987
179,5
392,5987

1435,2
1379,8256
0055,3744

0,5697631
0,5589030
0,0108601

5.
4,876902
0,123098

10,005
0,0098753
9,9951247

100000,
9,0000001
99990,9999999

MULTIPLICATION.

82. In *Multiplication* of Decimals, having placed the Factors (as in common Multiplication, 51.) you obſerve this general Rule, *viz. Cut off ſo many Places of Figures for Decimals in the Product as there contain'd Decimal Places in both the Factors.*

Thus Mult. 17,35
by 4
69,40

57,38
1,6
34428
5738
91,808

48,97
3,84
19588
39176
14691
188,0448

1750
0,76
10500
12250
1330.

0,758
0,52
1516
3790
0,39416

0,80507
0,305
402535
2415210
0,24554635

83. But if it happens that when the Operation is finiſhed, there are not ſo many Figures in the Product, as there are Places of Decimals in the Factors, then Cyphers are to be prefix'd to

the Product to make the Number of Places equal. As in theſe Examples.

,3	,09	,00175	,001
,2	,08	,0076	,01
,06	,0072	1050	,00001
		1225	
		,000013300	

59.	,1000101	1000000.
,000006	,1	,00000001
,000354	,01000101	,01

DIVISION.

84. Diviſion of Decimals is alſo perform'd by one general Rule, *viz. Divide the Numbers as if they were all Integers, and then cut off ſo many Figures for Decimals in the Quotient as when added to thoſe of the Diviſor, do make the ſame Number of Decimals as are contain'd in the Dividend.* Examples follow. (See 51, 54.)

1,6) 91,808 (57,38
80
11 8
11 2
.. 60
48
128
128
...

48,97) 188,0448 (3,84
146 91
41 134
39 176
. 1 9588
1 9588
....

1750) 1330,00 (,76
1225 0
. 105 00
105 00
.....

,52) ,39416 (,758
364
301
260
. 416
416
...

85. If, when the Diviſion is finiſhed, there are not ſo many Places in the Quotient as with thoſe Decimals in the Diviſor will equal the Number of Decimals in the Dividend, then Cyphers muſt be prefixed to equal that Number. As in theſe Examples.

```
,09),0072(,08          59),000354(,000006
      72                     354
     ——                      ———
      ..                     ...
```

86. When the Dividend is an integral Number, ſo many Cyphers are to be annexed thereto as there are decimal Places in the Diviſor; and the Quotient is in this Caſe integral.

```
Thus 2,5 ) 425.0 ( 170.          ,39) 741.00 ( 1900.
           25                         39
           ——                         ——
           175                        351
           175                        351
           ———                        ———
           ...0                       ...00

5,116) 1279.000 ( 250.           ,001 ) 597,000 ( 597000
       1023 2                           597
       ——————                           ———
        255 80                          ...
        255 80
        ——————
            .0
```

87. But if, when the Diviſion is thus far finiſhed, there happens to be any Remainder; then more Cyphers are to be annexed to the former in the Dividend, and ſo many Places are Decimal in the Quotient, as is evident in the following Examples. (See 61.)

,7) 673.0|00 (961,42
63
43
42
10
7
30
28
20
14
6

1,3) 12976.0|000 (9981,538
117
127
117
106
104
20
13
70
65
50
39
110
104
6

71) 29754,00|000 (41907,042
284
135
71
644
639
500
497
300
284
160
142
18

,015) 1.000|00 (66,66
90
100
90
100
90
100
90
10

88. N. B. In Caſe any large decimal Numbers are to be multiplied by each other, and it be required to retain only a certain umber of decimal Places in the Product, you may contract the ork and ſhorten the Labour by proceeding as follows, viz. *rite down the Multiplicand as uſual, and then write under it the ultiplier inverted, with the Unit's Place thereof under that Place the Multiplicand, whoſe Place you intend the Product ſhall extend ; then multiply, as uſual, by each Figure of the Multiplier, beginning*

ginning at thoſe of the Multiplicand which ſtand over it, neglecting thoſe to the Left, unleſs ſo far as to obſerve what would ariſe from multiplying the Figure immediately foregoing, which muſt be taken in at the Beginning of each Line, the firſt Figure of all which muſt ſtand under one another on the Right-hand.

89. Thus let it be required to multiply 3,141592 by 52,7438 to have four Places of Decimals in the Product. And alſo 104226,8672 by ,261799388; the Work for each will ſtand as below.

Multiplicand	3,1415 92	104226,867 2
Multiplier inverted	8347,25	883997 162,0
	1 5707 96	20845 373 4
	628 32	6253 612 0
	219 91	104 226 8
	12 57	72 958 7
	94	9 380 3
	25	937 9
		31 2
		8 3
		8
Product =	165,69 95	27286,529 4

90. The Reaſon of this Contraction will eaſily occur to any one who conſiders the Work at large, as it ſtands below for the firſt of theſe Examples.

3,1 4 1592
5 2,7438

2 5	1 32736
9 4	2 4776
125 6	6 368
2 199 1	1 44
6 283 1	8 4
157 079 6	0
165,699 5	0 01296

The perpendicular Line here drawn among the Figures cuts off all the ſuperfluous Part of the Work to the Right, and leaves the ſignificant Part on the Left, which is the ſame as the contracted Part, but in an inverted Order, which is the Reaſon why the Multiplier is inverted in that Caſe to produce it.

N. B. What relates to the Doctrine of *Repetends* or *circulating Decimals*, we ſhall refer to *Logarithms*; as this intricate Affair is moſt eaſily manageable by thoſe *artificial Numbers*.

CHAP.

CHAP. VIII.

The Reduction of Vulgar Fractions, and Quantities of divers Species to Decimal Numbers.

91. A Common Fraction is reduced into a decimal Expreſſion of the ſame Value by dividing the Numerator by the Denominator by the Rules in the preceding Chapter. As in the Examples below.

$$\begin{array}{l} \frac{2}{5} = 5\ )\ 2{,}0\ (= 0{,}4 \\ \qquad\quad\ \ 2\ 0 \\ \qquad\quad\ \ \overline{\cdot\ \cdot} \end{array}$$

$$\begin{array}{l} \frac{3}{8} = 8\ )\ 3{,}000\ (= 0{,}375 \\ \qquad\quad\ \ 24 \\ \qquad\quad\ \ \overline{\ \ 60} \\ \qquad\quad\ \ \ \ 56 \\ \qquad\quad\ \ \ \ \overline{\ \ 40} \\ \qquad\quad\ \ \ \ \ \ 40 \\ \qquad\quad\ \ \ \ \ \ \overline{\cdot\ \cdot} \end{array}$$

$$\begin{array}{l} \frac{1}{6} = 6\ )\ 1{,}000\ (= 0{,}16 \\ \qquad\quad\ \ 6 \\ \qquad\quad\ \ \overline{\ \ 40} \\ \qquad\quad\ \ \ \ 36 \\ \qquad\quad\ \ \ \ \overline{\ \cdot 4} \end{array}$$

$$\begin{array}{l} 5\,\frac{3}{8} = \frac{43}{8} = 8\ )\ 43{,}000\ (\ 5{,}375 \\ \qquad\qquad\qquad\ \ 40 \\ \qquad\qquad\qquad\ \ \overline{\ \ 30} \\ \qquad\qquad\qquad\ \ \ \ 24 \\ \qquad\qquad\qquad\ \ \ \ \overline{\ \ 60} \\ \qquad\qquad\qquad\ \ \ \ \ \ 56 \\ \qquad\qquad\qquad\ \ \ \ \ \ \overline{\ \ 40} \\ \qquad\qquad\qquad\ \ \ \ \ \ \ \ 40 \\ \qquad\qquad\qquad\ \ \ \ \ \ \ \ \overline{\cdot\ \cdot} \end{array}$$

92. By this Means you find the decimal Value of any Species of *Money*, *Weight*, *Meaſure*, *Time*, *Motion*, &c. in any Denomination above it. *viz.* By dividing Unity by the Number expreſſing how many of that Species make One in the ſuperior De-

Denomination propoſed. Thus becauſe four Farthings make one Penny; therefore $\frac{1}{4} = 0{,}25$ is the Decimal of a Penny for one Farthing. In like Manner, $\frac{1}{12} = 0{,}083 =$ the decimal Part of one Shilling for a Penny; and $\frac{1}{20} = 0{,}05 =$ the Decimal of a Pound for one Shilling.

93. Again, if it be required to know what decimal Part of a Pound one Farthing is, becauſe 960 Fathings make a Pound, therefore $\frac{1}{960} = 0{,}0010416$ is the Decimal of a Pound for one Farthing. After this Manner the following Tables are made, ſhewing the decimal Value for each Species of the integral Quantity or higheſt Denomination, which is always Unity, or 1, in the ſeveral Sorts of Money, Weight, Meaſure, *&c.*

One Pound	1.
Shilling	0,05
Penny	0,00416
Farthing	0,0010416

Troy Pound	1.
Ounce	0,083
Dram	0,010416
Scruple	0,003472
Grain	0,0001736

One Tun	1.
Hundred	0,05
Quarter	0,0125
Pound	0,000446
Ounce	0,000028

One Mile	1.
Furlong	0,125
Rod	0,003125
Foot	0,000189
Inch	0,000016

One Day	1.
Hour	0,0416
Minute	0,000694
Second	0,0000011

A Circle	1.
Sign	0,083
Degree	0,0027
Minute	0,00004
Second	0,0000006

One Load	1.
Quarter	0,2
Buſhel	0,025
Gallon	0,003125
Pint	0,00039

One Tun of Wine	1.
Pipe	0,5
Hogſhead	0,25
Tierce	0,16
Gallon	0,003967.

94. Having any Quantity expreſſed in divers Species, as 15*l.* 17*s.* 4*d.* 3*f.* 'tis eaſy, by the Tables, to turn it into Decimals.

$$\text{Thus}\begin{cases} 1 \times 15 = 15\,l. \\ 0{,}05 \times 17 = 0{,}85 \\ 0{,}00416 \times 4 = 0{,}016 \\ 0{,}001 \times 3 = 0{,}003 \end{cases}$$

Therefore 15*l.* 17*s.* 4*d.* 3*f.* = £ 15,869

95. Hence, by this Method, it will be eaſy to perform ſome Operations in Arithmetic, which would otherwiſe prove very irkſome and difficult. Thus, ſuppoſe it was required to

Feet Inches
Multiply 12 : $9\frac{3}{4}$ = 12,8124
By 7 : $5\frac{1}{2}$ = 7,4582

Now this would greatly puzzle a Learner to perform in the common Way, but when reduced to Decimals, is only a Caſe of common Multiplication. And it ſhould be a Maxim with Schoolmaſters never to torture the Genius of a Scholar with things extremely difficult, and at the ſame Time unneceſſary, or performable by eaſier Methods; which is but too commonly the Caſe.

CHAP. IX.

Of the Ratios of Numbers, and the Rules of Proportion, or Rule of Three Direct and Inverſe.

96. AS the Ratio of Numbers conſiſts of a Compariſon or Relation in reſpect of Magnitude; and as one Number may exceed another both by *Addition* and *Multiplication*, therefore two Sorts of Ratios will ariſe in the Compariſon of Numbers, *viz.* one, which will be expreſſed by their Difference when the leſſer is taken from the greater; and another, which will be expreſſed by the Quotient, in dividing the greater by the leſſer.

97. Hence, in this Series of Numbers 1, 2, 3, 4, 5, 6, 7, 8, &c. where each is greater or leſs than the other by Addition or Subtraction of 1, the common Difference will be 1 between any two contiguous Terms, which is therefore called the *common*

Ratio of the Series; thus the Ratio of 2 to 1, is $2 - 1 = 1$; of 5 to 4, is $5 - 4 = 1$; and ſo of the Reſt. But the Ratio of 3 to 1, is $3 - 1 = 2$, and 5 to $3 = 5 - 3 = 2$, which is double the former. And the Ratio of 4 to $1 = 4 - 1 = 3$, and 5 to $2 = 5 - 2 = 3$, *&c.* which are triple the Firſt. In the following Series 1, 3, 5, 7, 9, 11, 13, 15, *&c.* the Ratio or common Difference is 2. In this Series 5, 9, 13, 17, 21, 25, *&c.* the Ratio is 4; and ſo it may be any given Number, by which any Series of Numbers encreaſe or decreaſe. And hence ſuch a Series of Numbers are ſaid to be in *arithmetical Progreſſion.*

98. The other Sort of Ratio is that between Numbers which differ by a *common Multiplication or Diviſion*; Thus, if 1 and each ſubſequent Product be conſtantly multiplied by 2 this Series will ariſe, *viz.* 1, 2, 4, 8, 16, 32, 64, *&c.* and this conſtant Multiplier is the *common Ratio of the Series*; for the Ratio of any two proximate Numbers is the ſame; thus the Ratio of $2 : 1$ is $\frac{2}{1} = 2$; and the Ratio of $16 : 8 = \frac{16}{8} = 2$. Again, in this Series 1. 3. 9. 27, 81, 243, *&c.* the Ratio is 3; thus $3 : 1 = \frac{3}{1} = 3$, or $243 : 81 = \frac{243}{81} = 3$, and ſo of others.

99. In theſe Series, the Ratio of the third Term to the firſt is not *double* that of the Second to the Firſt, (as in arithmetical Series) but is ſaid to be *duplicate of it*; thus $4 : 1 = 2 \times 2 = 4$, whereas $2 : 1 = 2$, in the firſt Series; and in the ſecond Series, $9 : 1 = 3 \times 3 = 9$, whereas $3 : 1 = 3$. Again, the Ratio of the fourth Term to the Firſt is not *triple* (as in 91) but *triplicate* of the Ratio of the ſecond to the firſt. Thus, $8 : 1 = 2 \times 2 \times 2 = 8$, which is triplicate of $2 : 1 = 2$; or this latter Ratio is three Times involved in the former. Thus alſo $27 : 1 = 3 \times 3 \times 3 = 27$, which therefore is *triplicate* the Ratio of $3 : 1 = 3$, Whence a Series of Numbers having ſuch a Ratio is ſaid to be in *geometrical Progreſſion.*

100. Hence it is evident, that in order to make any Ratio twice as great as before, it muſt be multiplied by itſelf. Thus $2 : 1$ added to $2 : 1$, is $\frac{2}{1} \times \frac{2}{1} = \frac{4}{1}$; or $\frac{3}{1} \times \frac{3}{1} = \frac{9}{1}$, is $3 : 1$ added to it ſelf, or made twice as great. Conſequently, the *Addition of geometrical Ratios* is performed by multiplying thoſe Ratios by each other. Thus the Sum of the Ratios $5 : 3$ and $8 : 7$ is $\frac{5}{3} \times \frac{8}{7}$

=

$=\frac{40}{21}=40:21$. Again, $1:3$ added to $5:9$ makes the Sum equal to the Ratio $5:27$, becauſe $\frac{1}{3}\times\frac{5}{9}=\frac{5}{27}$.

101. On the other Hand, the *Subtraction of the geometrical Ratios* is performed by dividing the greater by the leſſer. Thus, if from $4:1$ I take $2:1$, there will remain $2:1$, becauſe $\frac{2}{1})\frac{4}{1}(=\frac{2}{1}$. In like Manner, if from the Ratio $40:21$ I ſubduct the Ratio of $8:7$, there will remain the Ratio of $5:3$ for $\frac{8}{7})\frac{40}{21}(=\frac{5}{3}$. And from the Ratio $5:27$ if we take $5:9$, there will remain $1:3$; becauſe $\frac{5}{9})\frac{5}{27}(=\frac{1}{3}$.

102. From a Compariſon of geometrical Ratios reſults the *Doctrine of Proportion or Analogy* (21) for if there be three Numbers, ſuch that the Ratio between the firſt and ſecond be the ſame with the Ratio between the ſecond and third, then are thoſe Numbers ſaid to be *Proportionals*, as 1, 2, 4; for $1:2=2:4$, or $1:2::2:4$. Alſo 1, 3, 9, are *proportional*, becauſe $1:3=3:9$ (by 98.) Thus alſo four Numbers are proportional, when the Ratio is the ſame between the firſt and ſecond, as it is between the third and fourth; as 1, 2, 4, 8, or 1, 3, 9, 27; becauſe $1:2=4:8$; and $1:3=9:27$.

103. Theſe proportional Numbers are ſuch as either ſucceed each other immediately in the Series, as $1:2::4:8$, or $4:8::16:32$; and the Proportion is ſaid to be *continued*, and ſuch Terms are called *continual Proportionals*. But if the Ratios are taken between ſuch Pairs of Numbers as do not ſtand together, or immediately follow each other in the Series, then is the Proportion ſaid to be *diſcontinued* or *disjunct*. As in theſe $1:2::8:16$, or $4:8::32:64$; or $1:3::81:243$. And this makes what is vulgarly called the *Golden Rule* (becauſe of its Uſefulneſs) or *Rule of Three*.

104. For by this Rule, if any three Numbers are given as 3, 9, 81, a Fourth may be found which ſhall be in Proportion, that is, ſhall have the ſame Ratio to the third Term 81, as there is between the two firſt 3 and 9. And becauſe this fourth Term is as the unknown or ſought, let us call it x. Then by Suppoſition, $3:9::81:x$; therefore $\frac{9}{3}=\frac{x}{81}$ (by the Nature of the Series 98.)

105. Now it is an *Axiom, that if equal Things are multiplied by equal Things, the Products will be equal*; therefore if the two equal Ratios $\frac{9}{3}$, $\frac{x}{81}$ be each multiplied by the ſame Number 81, we ſhall have $\frac{9\times 81}{3}=\frac{x\times 81}{81}=x$, as is evident becauſe $\frac{81}{81}=1$, and ſo makes no Alteration in the Value of x; therefore the Rule is, *multiply the ſecond and third Numbers together, and divide the Product by the Firſt, the Quotient will be the fourth Number ſought.* Thus $\frac{9\times 81}{3}=243$, ſo that 3 : 9 :: 81 : 243, according to (98.)

106. Hence this Rule comes to be of very great and frequent Uſe in the various practical Affairs of Life, which I ſhall exemplify by a few Queſtions, as follows.

If 3 Yards coſt 9 Shillings, what will 81 Yards coſt? See the Operation.

```
Yds. S.  Yds.   £.   S.
 3 : 9 :: 81 :  12   3
           9
          ——    20
     3 ) 729 (  243 ( 12l.
         6      20
         —      ——
         12     43
         12     40
         —      —
         ··9    ·3         Anſwer 12l. 3s.
           9
           —
           ·
```

If 100*l.* gain 5*l.* Intereſt, what will 750*l.* gain in the ſame Time?

```
  l.    l.     l.    l.   s.
 100 :  5 :: 750 :  37   10
               5
            ——————
100 ) 3750,0 ( 37,5 = 37l. 10s.
      3750 0
      ——————
       ····
```

If the Moon deſcribes the whole Ecliptic, or 360° in $27\frac{1}{2}$ Days, how many Degrees does ſhe paſs thro' in one Day?

$$\begin{array}{lllll} & & \text{D.} & & \text{D.} \\ \textit{Analogy.} & 27{,}5 : & 360^\circ & :: 1 : & 13^\circ{,}09 \\ & & 1 & & \end{array}$$

$$\begin{array}{rl} 27{,}5\ )\ 360\ (\ 13{,}09, & \textit{Anſwer.} \\ \underline{275} & \\ 850 & \\ \underline{825} & \\ 2500 & \\ \underline{2475} & \\ ..35 & \end{array}$$

According to the accurate Meaſures of the *French*, there are 57060 *Toiſes* in a Degree, or 242360 *Paris* Feet; the Circumference therefore of a great Circle is 123249600 Feet; and the *Paris* Foot is to the *Engliſh*, as 1068 to 1000; Quere how many *Engliſh* Feet and Miles are in the Earth's Circumference?

$$\begin{array}{llll} & & \textit{Paris Feet.} & \textit{Eng. Feet.} \\ \text{Analogy } 1000 : & 1068 :: & 123249600 : & 131630573 \end{array}$$

Then becauſe 5280 Feet = 1 Mile, ſay;

$$\begin{array}{llllll} \textit{F.} & \textit{M.} & \textit{F.} & \textit{M.} & \textit{Yds.} & \textit{F.} \\ \text{As } 5280 : & 1 :: & 131630573 : & 24930 & 57 & 2. \end{array}$$

107. When the Ratio of the Series is carried on by Diviſion below the firſt Term, as it is above it by Multiplication, as thus, $16 : 8 : 4 : 2 : 1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16}$, &c. then are the Ratios of theſe fractional Numbers to Unity, ſaid to be *inverſly* or *reciprocally* as the Ratios of the integral Numbers to Unity; that is, the Ratio of $\frac{1}{8}$ to 1 is the Reciprocal of 8 to 1; or $\frac{1}{8}$ is as *much leſs* as 8 is *greater* than Unity or 1.

108. When, therefore, any Queſtion in the Rule of Three is propoſed, and is of ſuch a Nature, that the fourth Number x, or Conſequent of the ſecond Ratio of the Analogy is reciprocally to its Antecedent of what the Conſequent in the firſt Ratio is to its Antecedent, then the *Rule of Three* is ſaid to be *Inverſe*.

And the Terms muſt be ſtated in a contrary or inverſe Order, as in the following Queſtion.

If 12 *Men do a Piece of Work in* 15 *Days, in how many Days will* 20 *Men do the ſame?*

Here it is evident, the 4th Number x cannot ſtand in the 4th Place, as before (104) *viz.* $12 : 15 :: 20 : x$; for then x would be directly to 20, as 15 to 12; but x muſt be leſs than 20, and therefore reciprocally as 15 to 12. Again it is plain, that $\overset{M.}{20} : \overset{M.}{12} :: \overset{D.}{15} : x$ Days; and becauſe the Proportion is here direct, therefore $\frac{12 \times 15}{20} = x$ (105.) Conſequently if we take the Terms as they ſtand in Queſtions of this Sort, the Rule for operating them in this, *Multiply the firſt and ſecond Numbers together, and divide by the Third, and the Quotient will be the fourth Number* (x) *ſought.* So in the preſent Caſe $\frac{12 \times 15}{20} = x = 9$ Days, the Anſwer.

109. I ſhall here ſubjoin a Queſtion or two of this Sort, as follows.

A Friend lends me 372l. *for* 7 *Years and* 8 *Months, how long muſt I lend him* 496l. *for an Equivalent?*

Then per Rule $\frac{496 \times 7{,}6}{372} = 5{,}75$ Years, Anſwer.

If 3 *Men, and* 4 *Women, can do a Piece of Work in* 56 *Days, how long will one Man and one Woman be doing the ſame?*

Becauſe of 3 Men and 4 Women, ſome Number muſt be found that may be divided by 3 and 4 without a Remainder, as the Number 12; then make the 3 Men or 4 Women equal to 12 Boys; and 1 Man will be equal to 4 Boys, and 1 Woman to 3 Boys, and 1 Man and 1 Woman to 7 Boys; then the Queſtion is reduced to this, *If* 12 *Boys do a Piece of Work in* 56 *Days, in how many Days will* 7 *Boys do the ſame?*

Anſwer $\frac{12 \times 56}{7} = 96$ Days.

110. I ſhall here ſay Nothing of the *compound Rule of Proportion*, or, as it is uſually called, the *Double Rule of Three*, wherein 5 Numbers are given to find a Sixth by means of two Analogies,

gies, because this will be beſt explained, and the Reaſon of Operation will be more evidently ſeen, in the Method of treating this Subject in Algebra.

111. From what has been ſaid of Ratios, it is eaſy to make a Compariſon of their Magnitude, having firſt premiſed this Definition, viz. *That Ratio is ſaid to be* greater, equal to, or leſs *than another, whoſe Antecedent hath* a greater, or an equal, or a leſs *Proportion to its Conſequent; than the other's Antecedent hath to its Conſequent.* Thus the Ratio of 6 to 3 is ſaid to be greater, and the Ratio of 4 to 3 leſs than the Ratio of 5 to 3. Thus again, the Ratio $6 : 3$ is greater, and the Ratio $6 : 5$ leſs than the Ratio of $6 : 4$.

112. Hence, when two Ratios are to be compared whoſe Antecedent and Conſequent are both different, *it will be proper to reduce them to the ſame Antecedent, or the ſame Conſequent, before the Compariſon be made*; as for Inſtance, ſuppoſe I would know which of the two Ratios $7 : 5$ or $4 : 3$ be the greater; to know this, I ſay, as $4 : 3 :: 7 : 5\frac{1}{4}$; then it is evident, $7 : 5$ is a greater Ratio than $7 : 5\frac{1}{4}$ (by 111,) and conſequently greater than $4 : 3$. Again, ſuppoſe I would compare the Ratios $3 : 4$ and $5 : 7$; then I ſay, as $3 : 4 :: 5 : \frac{20}{3} = 7 - \frac{1}{3}$; but the Ratio of $5 : 7 - \frac{1}{3}$ is greater than the Ratio of $5 : 7$, and therefore the Ratio $3 : 4$ is greater than the Ratio of $5 : 7$.

113. In any Series of Numbers, 48, 40, 30, 15, *the Ratio of the Extremes is ſaid to be compounded of all the intermediate Ratios*; viz. $48 : 15 = 48 : 40 + 40 : 30 + 30 : 15$; which will eaſily appear by placing all Fraction-wiſe, thus; $\frac{48}{15} = \frac{48}{40} \times \frac{40}{30} \times \frac{30}{15} = \dfrac{48 \times 40 \times 30}{15 \times 40 \times 30}$, for it is plain, ſince 40×30 is in the Numerator and Denominator both, it makes no Alteration in the Value of the Fraction, which therefore is equal to $\frac{48}{15}$.

114. Hence, on the Contrary, any Ratio $48 : 15$ may be reſolved into any Number of other leſſer Ratios of which it doth conſiſt, as ſo many Parts of the Whole. Thus $\frac{48}{15} = \frac{48}{40} \times \frac{40}{30} \times \frac{30}{15}$; or $\frac{48}{15} = \frac{48}{45} \times \frac{45}{35} \times \frac{35}{15}$. Thus alſo the Ratio of 48 to 5 may be decompounded, or reſolved into any other Number of Ratios, as $\frac{48}{42} \times \frac{42}{36} \times \frac{36}{27} \times \frac{27}{13} \times \frac{13}{5}$, and ſo in any other Caſe.

115. In any two Quantities or Numbers, whoſe Difference is very ſmall in reſpect of the Quantities themſelves, if ſo much be *added to one*, and *ſubtracted from the other*, as ſhall make their *Difference double* or *triple*, or *half*, or a *third Part* of what it was before, then thoſe Quantities or Numbers ſhall be in a *Duplicate* or a *Triplicate*, or a *Subduplicate* or a *Subtriplicate Ratio* of that they were in before any ſuch Change was made, nearly.

116. Thus let there be two Numbers 10 and 11, whoſe Difference is 1, then if $\frac{1}{2}$ be added to 11 and taken from 10 we have $11\frac{1}{2}$ and $9\frac{1}{2}$, whoſe Difference is 2, double of the former Difference. Now I ſay, the Ratio of $10\frac{1}{2}$ to $9\frac{1}{2}$ is *duplicate* of that of 11 to 10 nearly; for the Ratio of $11\frac{1}{2}$ to $9\frac{1}{2}$ is reſolvable into the Ratios $11\frac{1}{2} : 10\frac{1}{2}$, and $10\frac{1}{2} : 9\frac{1}{2}$ (by 114.) Now the Ratio of $11\frac{1}{2} : 10\frac{1}{2}$ is greater than the Ratio 11 to 10, and the Ratio of $10\frac{1}{2} : 9\frac{1}{2}$ is nearly as much leſs (as will appear from 112) therefore the Sum of both thoſe Ratios will be nearly equal to twice the Ratio of 11 to 10, that is, $\frac{11\frac{1}{2}}{10\frac{1}{2}} \times \frac{10\frac{1}{2}}{9\frac{1}{2}} = \frac{11}{10} \times \frac{11}{10}$, nearly; for the firſt is $\frac{120,75}{99,75}$, and the latter is $\frac{121}{100}$, which are very nearly equal.

117. Again, if we add 1 to 11 and take it from 10, we ſhall have 12 and 9, whoſe Difference is 3; then will the Ratio 12 : 9 be triplicate, or three Times as great as the Ratio 11 : 10. For $\frac{12}{9} = \frac{12}{11} \times \frac{11}{10} \times \frac{10}{9}$ (by 114) $= \frac{1320}{990}$; and $\frac{11}{10} \times \frac{11}{10} \times \frac{11}{10} = \frac{1331}{1000}$, which two Fractions are very nearly equal.

118. Thirdly, if the Difference between 11 and 10 be reduced to half or a third Part, the Ratio will be reduced *ſubduplicately* or *ſubtriplicately*; thus add $\frac{1}{4}$ to 10, and take it from 11, and we have $10\frac{1}{4}$ and $10\frac{3}{4}$, whoſe Difference is half of the former. Now $10\frac{1}{4} : 10\frac{3}{4}$ is *ſubduplicate* of the Ratio 10 : 11, or as $\sqrt{10} : \sqrt{11}$. Alſo, if $\frac{1}{3}$ be added to 10 and taken from 11, you have $10\frac{1}{3}$ and $10\frac{2}{3}$, which are nearly in a *ſubtriplicate* Ratio of 10 to 11 or $\sqrt[3]{10}$ to $\sqrt[3]{11}$.

We ſee in theſe Examples how near theſe Ratios come to the Truth, where the Difference is no leſs than a 10th or 11th Part of the Whole; but if we ſuppoſe the Difference to be a much leſs Part of the Whole, as an 100th, a 1000th, *&c.* they will be much more accurate; ſo that to multiply or divide the Ratio it will be ſufficient to encreaſe or diminiſh one of the Numbers only. Thus 100 : 102 is *duplicate* of the Ratio 100 : 101; and 100 : 103 is *triplicate* the Ratio of 100 : 101. Alſo, $100 : 100\frac{1}{2}$ is *ſubduplicate*, and $100 : 100\frac{1}{3}$ is *ſubtriplicate* of the Ratio of 100 : 101, nearly.

119. Hence we obſerve, that the Ratio 100 : 101 is to the Ratio 100 : 102 as 1 : 2; and the Ratio 100 : 101 is to the Ratio 100 : 103, as 1 : 3; the Ratio $\overline{100 : 101} : \overline{100 : 104} :: 1 : 4$; and ſo on univerſally, which Theorem is of very great Uſe and ought to be well remembered by the *mathematical Reader*. What further relates to the Doctrine of *Ratios and Proportion*, will be delivered in Species (in the *Algebraic Part*) which will afford a more abſolute and univerſal Speculation of the Nature and Properties thereof than can be obtained from Numbers.

CHAP. X.

Of the Powers of Numbers, and the Extraction of Roots.

120. WHEN any Number is multiplied by itſelf, it is ſaid to be *ſquared*, and the Product is called the *Square* or ſecond Power of that Number; thus $2 \times 2 = 4$, ſo 4 is the Square of 2; and 2 is ſaid to be the *Square Root* of 4; and is thus expreſſed, $\sqrt{4} = 2$. So $7 \times 7 = 49$, and the Square Root of 49 is $\sqrt{49} = 7$, and ſo of any other Numbers.

121. As a Number multiplied by itſelf produces the Square, ſo that Square being multiplied by the ſaid Number, or Root, produces the *Cube*, or third Power. Thus $4 \times 2 = 2 \times 2 \times 2 = 8$, the *Cube* of 2; and $49 \times 7 = 7 \times 7 \times 7 = 343$, the *Cube* of 7. So the *Cube Root* of 8 is $\sqrt[3]{8} = 2$; and the *Cube Root* of 343 is $\sqrt[3]{343} = 7$; and ſo of others.

122. Thus you proceed by a conſtant Multiplication of every ſubſequent Product by the ſame Number to raiſe any Power of that Number you think fit, as you ſee done for the 9 Digits in the following Table to the ſixth Power.

Root, or firſt Power.	1	2	3	4	5	6	7	8	9
Square, or ſecond Power.	1	4	9	16	25	36	49	64	81
Cube, or third Power.	1	8	27	64	125	216	343	512	729
Biquadrate, or fifth Power.	1	16	81	256	625	1296	2401	4096	6561
Surſolid, or ſixth Power.	1	32	343	1024	3125	7776	16807	32768	59049

123. When the *Square Root* of any Number is propoſed to be found or extracted, it is done by the following

RULE.

Firſt, let the Figures of the Number be diſtinguiſhed into Pairs by fixing a Point over every other one, beginning at Unit's Place. Then write ſuch a Figure for the firſt Place in the Root, whoſe Square ſhall be equal to, or next leſs than the Figure or Figures from the laſt Point on the left Hand; then ſubtracting that Square, the other Figures of the Root will be found by taking down each Pair of Figures ſucceſſively to the Remainders, for new Dividends, and doubling the Root ſo far as extracted for the firſt Part of the Diviſor, enquire how often it is contained in the new Dividend, and place the Quotient for another Figure in the Root, and alſo annex it to the Diviſor, which will be then compleated.

124. This will be illuſtrated by the following Examples, Quere the Square Root of 144?

```
          · ·
          144 ( 12, the Square Root required,
          1
          —
    22 ) ·44
          44
          —
          ·,
```

The Reaſon of pointing the Number in this Manner is, becauſe there are always *as many Places of Figures in the Root as there are Points over the given Number*. But, till we come to Algebra, the Reaſon of the whole Operation will not ſo well appear as by working it at large (for the common Way is but a ſort of Contraction) in the Manner following.

$\dot{1}4\dot{4}$ (10
100 ($\frac{2}{20}$

20) 44 (2, then 10 + 2 = 12, the Root.
$\frac{2}{22}$) 44
..

Extract the Root of 219024.

$2\dot{1}9\dot{0}2\dot{4}$ (400 + 60 + 8 = 468, the Root.
160000

800) 59024 (60
$\frac{60}{860}$) 51600

460 × 2 = 920) 7424 (8
$\frac{8}{928}$) 7424
....

But this Example contracted in the comman Way ſtands thus.

$2\dot{1}9\dot{0}2\dot{4}$ (468, the Root.
16

86) 590
516

928) 7424
7424
....

Another Example here follows. To extract the Square Root of the Number 29506624.

```
                 · · · ·
                29506624 ( 5432 = the Root required.
                25
                —
       104 ) ·450
               416
               ———
      1083 ) ·3466
               3249
               ————
     10862 )  21724
              21724
              —————
              .....
```

125. It is plain that Number muſt be a Square, whoſe Root may be extracted without a *Remainder*; and ſuch Numbers may be as well *Decimal* as *Integral*, as in the following Example. What is the Square Root of 156,25.

```
             ·  ·  ·
            156,25 ( 12,5 the Root required.
            1
            —
     22 ) ·56
           44
           ——
    245 ) 1225
          1225
          ————
          ....
```

What is the Square Root of 50,2681 ?

```
            ·   · ·
            50,2681 ( 7,09 = Root required.
            49
            —
    1409 )  12681
            12681
            —————
            .....
```

Required the Square Root of 0,366025.

. . . .
0,366025 (0,605 = the Root.
0
—
36
36
—
1205) . .6025
6025
—
. . . .

What is the Square Root of 0,00015625?

. . . . .
0,00015625 (0,0125 = the Root.
1
—
22) .56
44
—
245) 1225
1225
—
. . . .

126. The Number that is not a Square, or whoſe Root cannot be extracted, is ſaid to be *ſurd*, or *irrational*; but the Roots of ſuch ſurd Numbers may be approximated in Decimals as near the Truth as required. For Inſtance, let it be required to extract the Square Root of 2. Annex to the given Surd, as many Pairs of Cyphers as you would have decimal Places in the Root, Thus

```
          . . . . . .
          2,0000000000 ( 1,41421
          1
          ─
    24 )  1 00
             96
            ──
   281 )     400
             281
            ───
  2824 )   11900
           11296
           ─────
 28282 )    60400
            56564
            ─────
282841 )    383600
            282841
            ──────
            100759
```

Thus it appears the Square Root of 2 is 1,41421 true to five Places of Decimals; if a greater Degree of Accuracy be required, more Places may be obtained by dividing the Remainder 100759 in the common Way by the Diviſor 28284 (omitting the firſt Figure 1) as you ſee below.

```
28284 ) 100759 ( 3562
         84852
         ──────
         159070
         141420
         ──────
          176500
          169704
          ──────
           67960
           56568
           ─────
           11392
```

Therefore the Square Root of 2 is ſtill more truely 1,414213562.

127. The Square Root of any Vulgar Fraction is extracted by extracting the Root of the Numerator and Denominator for the fractional Root. Thus the Square Root of $\frac{25}{36}$ is $\frac{5}{6}$; for 5 is the Root for 25, and 6 the Root of 36, and $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$; and ſo the Square Root of $\frac{144}{256}$ is $\sqrt{\frac{144}{256}} = \frac{12}{16}$; and ſo you proceed for any other. But in ſome Caſes, the beſt Way will be to convert the *Vulgar* into a *Decimal Fraction* (by 91.) and ſo extract the Root in Decimals, as above taught (in 125, 126.)

128. The Extraction of the *Cube Root* will prove too difficult a Taſk in common Numbers; and as Nothing can be eaſier by Logarithms, I ſhall there ſhew the Method of doing it. Beſides, the Reaſon of the Thing cannot be ſhewn till the Reader comes to the Algebraic Part, where it will be evident enough.

129. The Biquadrate Root of any Number is eaſily had by extracting the Square Root of the given Number firſt, and then the Square Root of that Root: Thus, let it be required to extract the *Biquadrate Root* of 4857532416.

```
          .  .  .  .  .
         4857532416, ( 69696,
         36
         --
   129 ) 1257
         1161
         ----
  1386 )   9653
           8316
           ----
 13929 ) 133724
         125361
         ------
139386 )  836316
          836316
          ------
          ......
```

Then

```
            .  .  .
Then 69696, ( 264, = the Biquadrate Root.
     4
     —
46 ) 296
     276
     ———
524 ) 2096
      2096
      ————
      ....
```

For 264 × 264 × 264 × 264 = 4857532416. And thus you proceed in any other Caſe where the *Biquadrate Root* is required.

CHAP. XI.

Of the Nature and Uſe of LOGARITHMS.

130. IF a Series of Numbers in *Arithmetical Progreſſion*, beginning from 0, and whoſe common Ratio is Unity, be appoſitely placed over another Series of Numbers in *Geometrical Progreſſion*, in the following Manner, *viz.*

Arith.	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	*&c.*
Geom.	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	*&c.*

Then we may obſerve the following Things.

131. *Firſt*; The ſeveral Terms in the Arithmetical Series *expound the Ratio* of the correſponding Terms in the Geometrical Series to the firſt Term; thus, 2 in the upper Series ſhews the Ratio of the Geometrical Series is *twice* repeated between 4 and 1; 5 ſhews the Ratio 5 Times repeated between 32 and 1; or that the Ratio 32 : 1 is 5 Times as great as the Ratio 2 : 1; and 9 denotes the Ratio 512 : 1 to be 9 Times as great as the firſt Ratio 2 : 1; and ſo of the Reſt.

132. *Secondly*; That to the *Addition* of any two or more Terms in the Arithmetical Series, correſponds a *Multiplication* of the Numbers under them in the Geometrical Series; as in the following Examples.

A. $1 + 2 = 3$; $2 + 3 + 4 = 9$; $1 + 3 + 5 = 9$; *&c.*
G. $2 \times 4 = 8$; $4 \times 8 \times 16 = 512$; $2 \times 8 \times 32 = 512$; *&c.*

133. *Thirdly*; That for every *Subtraction* of Terms in the Arithmetical Series, there correſponds a *Diviſion* of the Numbers under them in the Geometrical Series; as in theſe Inſtances.

A. $2 - 1 = 1$; $7 - 3 = 4$; $9 - 4 = 5$; *&c.*
G. $4 \div 2 = 2$; $128 \div 8 = 16$; $512 \div 16 = 32$; *&c.*

134. *Fourthly*; That when any Term in the Arithmetical Series is *doubled*, *tripled*, *quadrupled*, &c. there anſwers an *Involution* of the correſponding Term in the Geometrical Series to the *ſecond*, *third*, *fourth*, &c. *Power*, Thus

A. $3 + 3 = 6$; $3 + 3 + 3 = 9$; $2 + 2 + 2 + 2 = 8$.
G. $8 \times 8 = 64$; $8 \times 8 \times 8 = 512$; $4 \times 4 \times 4 \times 4 = 256$.

135. *Fifthly*; That if any Term in the Arithmetical Series be *divided* by 2, 3, 4, *&c.* there anſwers an Evolution or Extraction of the *Square*, Cubic, Biquadratic, *&c.* Root in the correſponding Term of the Geometrical Series. Thus

A. $4 \div 2 = 2$; $6 \div 3 = 2$; $8 \div 4 = 2$; *&c.*
G. $\sqrt{16} = 4$; $\sqrt[3]{64} = 4$; $\sqrt[4]{256} = 4$; *&c.*

136. *Sixthly*; That in the Arithmetical Series, if any three proximate Numbers be taken, the *Sum of the two Extreams is equal to double the middle Number*; thus, 3, 4, 5, give $3 + 5 = 4 \times 2 = 8$; and 7, 8, 9 give $7 + 9 = 8 \times 2 = 16$. Whence alſo $\frac{3 + 5}{2} = 4$, and $\frac{7 + 9}{2} = 8$, or *half the Sum of the Extremes is equal to the Mean*. And the Caſe is the ſame if any two Numbers of the Series be taken, their Sum will be always double of the Mean or middle Term between them. Thus, 1, 3, 5 give $1 + 5 = 2 \times 3 = 6$; and 2, 5, 8 give $2 + 8 = 2 \times 5 = 10$. Alſo if 4 Terms be taken any how, the *Sum of the two Extremes will always be equal to the Sum of the two Means*; thus, 1, 2, 3, 4, give $1 + 4 = 2 + 3 = 5$; and 3, 5, 7, 9, give $3 + 9 = 5 + 7 = 12$, *&c.*

137. *Seventhly*; That as every *Addition* of Terms in the *Arithmetical Series* has a correſponding *Multiplication* of Terms in the *Geometrical Series* (by 132,) therefore *the Product of the two Extremes, in any three* Proportionals, *is equal to the Square of the Mean*; thus, in 8, 16, 32 we have $32 \times 8 = 16 \times 16 = 256$; and therefore alſo $\sqrt{32 \times 8} = 16$, *viz.* the *Square Root of the Product of any two Numbers is always a Mean Proportional between them.*

138. *Eighthly*; That of any Terms in the Geometrical Series, *the Product of the two Extremes is ever equal to the Product of the two Means.* Thus, 2, 4, 8, 16, give $16 \times 2 = 8 \times 4 = 32$. And 2, 8, 32, 128, give $128 \times 2 = 32 \times 8 = 256$.

139. *Ninthly*; Hence, of four Terms in the Arithmetical Series, the *laſt is equal to the Difference between the Sum of the two Means and the firſt Term.* Thus, of 3, 5, 7, 9, we have $5 + 7 - 3 = 9$; or $7 + 5 - 9 = 3$. And in the *Geometrical Series*, the *laſt of the four Proportionals is equal to the Product of the two Means divided by the firſt Term.* Thus, of 8 : 32 : 128 : 512, we have $\frac{32 \times 128}{8} = 512$; or $\frac{128 \times 32}{512} = 8$. As we have elſewhere ſhewn (in 105.)

140. From what has been hitherto premiſed, it appears, that the Series of Numbers in Arithmetical Progreſſion, are the LOGARITHMS of the Numbers in the other Series of Geometrical Proportionals; for all that we mean by *Logarithms*, is no more than *ſuch a Sort of Numbers as are artfully contrived to expreſs or expound the Ratios of common natural Numbers, conſidered as Terms in a Scale of Geometrical Proportion.* Now theſe Numbers in Arithmetical Progreſſion, anſwer every Part of this Definition of Logarithms with reſpect to the Series below them (by what was obſerved in 131.) and therefore are their *Logarithms*, i. e. *Exponents of their Ratios*, as the Word imports in its *Greek Etymology.*

141. From hence it appears, that if we conſider *our natural Numbers* as Terms in a Scale of Geometrical Progreſſion, then if ſuch other Numbers were invented and adapted thereto in a Series of Arithmetical Progreſſion, theſe would be *Logarithms*

of the other; ſo their *Addition* and *Subtraction* would anſwer a *Multiplication* and *Diviſion* of the reſpective common Numbers (by 132, 133.) Alſo by *doubling* or *tripling* them, we Square or Cube their Numbers; (by 134) or by dividing by 2, 3, *&c.* you extract the *Square, Cubic, &c.* Root of their Numbers, (by 135.)

142. Now ſuch a Table or Canon of *Logarithmic Numbers* has been contrived and compoſed by our late Mathematicians, and are in every one's Hands for Uſe. I ſuppoſe I ſcarce need tell the Reader that Lord *Napier* (a *Scotch* Nobleman) invented, and together with the Aſſiſtance of our Countryman Mr. *Henry Brigs*, calculated and compleated the Canon in preſent Uſe. The Labour of doing this was prodigious in the Way they took for it, but of late, eaſier and more conciſe Methods have been invented, of which more hereafter.*

143. That the Reader may have ſome Idea of the Method they took for this Purpoſe, he muſt conſider that 1, 10, 100, 1000, 10000, 100000, *&c.* are a Series of Numbers in Geometrical Progreſſion, whoſe common Ratio is 10; and a Series of Numbers in Arithmetical Progreſſion adapted to them as Logarithms will ſtand as below.

A.	0.	1.	2.	3.	4.	5.	*&c.*
G.	1.	10.	100.	1000.	10000.	100000.	*&c.*

144. If now we ſuppoſe the common Ratio 10 to be divided into 10000000 equal Parts, or *Ratiunculæ*, then ſince the Logarithm of 1 to 10 is 1, this Logarithm or Unity 1, will alſo be divided into 10000000 equal Parts, or *Decimals*; of which half the Number, *viz.* 0,5000000 will be the Logarithm of the Mean Proportional between 1 and 10, which let us call A.

Then will the Logarithms be for

Numb.		Logarithms.
1	:	0,0000000
A	:	0,5000000
10	:	1,0000000

---

* Several other Methods of conſtructing a Canon of Logarithms will be delivered when we come to treat of *Algebra* and *Fluxions*; this we have here given, ſtands firſt in Order, and riſes immediately from the Principles of *common Arithmetic*.

145. In like Manner, a Mean Proportional between A and 10, call B, and its Logarithm will be half the Sum of the Logarithms of A and 10; and ſo

The Logarithms will be for
A : 0,5000000
B : 0,7500000
10 : 1,0000000

146. Alſo between 1 and A, and A and B, you find Mean Proportionals and their Logarithms, and thus you may conceive the Proceſs for every one of the 10000000 Means; among which you'll find *eight Means* which will be ſo near the ſame with our eight Digits 2, 3, 4, 5, 6, 7, 8, 9, that the Difference will be wholly inconſiderable, and alſo their Logarithms, which will be as below, *viz.*

Thus,

Numb.		Logarithms.	Numb.		Logarithms.
1	:	0,0000000	6	:	0,7781513
2	:	0,3010300	7	:	0,8450980
3	:	0,4771213	8	:	0,9030900
4	:	0,6020600	9	:	0,9542425
5	:	0,6989700	10	:	1,0000000

147. That is, ſince there are 10000000 Mean Proportionals between 1 and 10, the Number 2 will be the 3010300th of theſe; ſo the Number 3 will be the 4771213th; the Number 5 will be the 6989700th, and ſo of the Reſt. Alſo the Ratio or Diſtance of 4 from Unity being twice as great as the Diſtance of 2, its Logarithm is twice as big, and for the ſame Reaſon the Logarithm of 9 is twice as big as the Logarithm of 3; and the Logarithm of 8, three Times as great as the Logarithm of 2; and ſo on.

148. In like Manner, the Logarithm for all the Numbers between 10 and 100, 100 and 1000, and ſo on to 100000, denote the Places or Diſtances of thoſe Numbers in a Scale of Geometrical Proportionals, conſiſting of 10000000, 20000000, 30000000, *&c.* Terms. Thus 73 is the 18633229th Term; 743 is the 28709888th Term; and 9745 is the 39887818th Term in the Scale or Series of 40000000.

149. But ſince we make the Exponent or Logarithm of 1 to 10, to be 1; that of 1 to 100, 2; *&c.* (by 143) therefore

theſe Logarithms muſt be looked upon as *Decimal Numbers*, and only the firſt Figure Integral; which is called the Index; and they are thus expreſſed.

viz.
3 : 0,4771213
73 : 1,8633229
743 : 2,8709888
9745 : 3,9887818

150. And here you obſerve the Index of the Logarithm is *leſs by Unity* than the Number of Figures in that Number of which it is the Logarithm, the Reaſon of which is very plain from (143) and what has been ſince delivered. N. B. *In reading, this Table of Logarithms ſhould lie before the Eye for Inſpection.*

151. If a Number conſiſt of the ſame Figures, whether it be *integral*, *mixed*, or *pure Decimals*, the Logarithm will ſtill be the ſame, except the *Index*, *which will be always leſs by* 1, *than the Number of Places in the Integral Part* (149). And when the Number is purely Decimal, the Indexes will alſo be expreſſed Decimally, all which will be clear by the Examples in the following Table.

N. B. As many as the Decimal Index is leſs than 9, ſo many Cyphers are prefixed to the Decimal Numbers, as is here evident.

Numbers	Logarithms.
9541	3,9795939
954,1	2,9795939
95,41	1,9795939
9,541	0,9795939
0,9541	,9,9795939
0,09541	,8,9795939
0,009541	,7,9795939

152. It remains now that we ſhew how commodiouſly the Operations of common Arithmetic are performed by Logarithms, and firſt,

Of MULTIPLICATION *by* LOGARITHMS.

The Rule for the Operation is this, *add together the Logarithms of the Factors, the Sum is the Logarithm of the Product.* (By 132) But the Difficulty conſiſts in finding the proper Indices to the Sums; for which obſerve the following Particulars. (1.) If the Indeces are both *integral*, the Sum is ſo too. (2.) If one be Integral, and the other Decimal, the Sum, if under 10, will be Decimal; if juſt 10, or more than 10, caſt away 10, an the Remainder is Integral. (3.) If the Indices are both Decimal, and the Sum above 10, caſt away 10, and the Remainde will be Decimal; and then the Cyphers to be prefixed to the Decimal Product, will be as many as ſuch an Index is leſs than 9. See the following Examples.

153. *Examples of* INTEGERS.

Example I. Multiply 12 = 1,079181
By 8 = 0,903090
The Product 96 = 1,982271

Example II. Multiply 526 = 2,720986
By 100 = 2,000000
The Product 52600 = 4,720986

Example III. Multiply 987600 = 5,994581
By 517 = 2,713490
The Product 510589200 = 8,708071

154. *Examples of mixed Numbers.*

Example I. Multiply 12,4 = 1,0934217
By 3,6 = 0,5563025
The Product 44,64 = 1,6497242

Example II. Multiply 36,5 = 1,5622929
By 0,00019 = ,6,2787536
The Product 0,006935 = ,7,8410465

Example III. Multiply 0,762 = ,9,8819550
By 570 = 2,7558748
The Product 434,34 = 2,6378298

Example IV. Multiply 0,0097 = ,7,9867717
By 0,00021 = ,6,3222193
The Product 0,000002037 = ,4,3089910

155. DIVISION of *Integers* and *Decimals* by LOGARITHMS.

The Rule. From the Logarithm of the *Dividend*,
Subtract the Logarithm of the *Diviſor*;
The *Remainder* is the Logarithm of the *Quotient*.

Example I. Divide 44,64 = 1,6497242
By 12,4 = 1,0934217
Quotient 3,6 = 0,5563025

Example II. Divide 310 = 2,4913617
By 4,275 = 0,6309361
Quotient 72,51457 = 1,8604256

Example III. Divide 434,34 = 2,6378298
By 7,62 = 9,8819550
Quotient 570 = 2,7558748

Example IV. Divide ,006935 = ,7,8410465
By 36,5 = 1,5622929
Quotient ,00019 = ,6,2787536

Example V. Divide ,0000002073 = ,4,3089910
By ,00021 = ,6,3222193
Quotient ,0097 = ,7,9867717

156. As the Operation of ſuch Decimal Numbers as contain *ingle* or *compound Repetends* is moſt eaſily performed by *Logaithms*, (ſee Inſt. 90) ſo we ſhall here proceed to that Buſineſs, aving firſt premiſed, that if one Number *A*, be to be divided by nother *B*, the Quotient will be the ſame as when Unity is divided y the Number *B*, and the Quotient multiplied by the other *A*; hus 4) 12 (3 = $\frac{1}{4}$ × 12; now if from the Logarithm of Uity or 0, you ſubtract the Logarithm of any Number, the Reainder is called the *Arithmetical Compliment* of that Number.

hus from the Logarithm of Unity = 0,0000000
ake the Logarithm of 12,4 = 1,0934217
emains the Arithmetical Compliment = ,8,9065783

hich is plainly nothing more than the Logarithm of the Fraction $\frac{1}{2,4}$; if therefore

To the Logarithm of 44,64 = 1,6497242
You add the Arithmetical Compliment of 12,4 = ,8,9065783
The Sum is the Logarithm of 3,6 = 0,5563025
The ſame as in Example I. of Diviſion (155).

157. Any Digit multiplied by 10 and divided by 9, become a *Repetend*; thus 6 × 10 = 60, and 9) 60 (= 6,666$\acute{6}$, *&c.* = 6,$\acute{6}$; and the ſame in Decimals 0,6 × 10 = 6, and 9) 6 (= 0,$\acute{6}$. Alſo any Number multiplied by Unit with as many Cyphers annexed as it contains Places, and then divided by as many Nines, becomes a *compound Repetend.* Thus 23 × 100 = 2300, and 99) 2300, (= 23,$\acute{2}\acute{3}$, and 527, × 1000, = 527000, and 999,) 527000, (= $\acute{5}2\acute{7}$; and ſo of others.

158. Hence ſince the Logarithms of $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, *&c.* are 0,0457575, 0,0043648, 0,0004345, *&c.* therefore we eaſily obtain the Logarithms of any pure, ſingle, or compound Repetend.

Example. Required the Logarithm of $\acute{6}$?
To the Logarithm in the Table for 6 = 0,7781512
Add the Logarithm of $\frac{10}{9}$ = 0,0457575
The Sum is the Logarithm of $\acute{6}$ = 0,8239087

And becauſe it may be of Uſe ſometimes, I have here ſubjoined a Table of the Logarithms of all the Nine Digits *perpetually circulating:*

$\acute{1}$ = 0,0457575
$\acute{2}$ = 0,3467875
$\acute{3}$ = 0,5228787
$\acute{4}$ = 0,6478175
$\acute{5}$ = 0,7447275
$\acute{6}$ = 0,8239087
$\acute{7}$ = 0,8908555
$\acute{8}$ = 0,9488475
$\acute{9}$ = 1,0000000

159. In the ſame Manner we proceed for the Logarithms of pure compound Repetends.

Example 1. Required the Logarithm of the Repetend $\acute{2}\acute{4}$?
To the Tabular Logarithm of 24, = 1,3802112
Add the Logarithm of $\frac{100}{99}$ = 0,0043648
The Sum is the Logarithm of $\acute{2}\acute{4}$, = 1,3845760

Example 2. Required the Logarithm of $\acute{3}6,\acute{5}$?
To the Tabular Logarithm of $\acute{3}6,\acute{5}$ = 1,5622929
Add the Logarithm of $\frac{1000}{999}$ = 0,0004345
The Sum is the Logarithm of $\acute{3}6,\acute{5}$ 1,5627274.

160. Now with Reſpect to *mixed Repetends*, as $2,\dot{6}$, $2,7\dot{5}\dot{3}$, $7\dot{2}5,\dot{6}$, *&c.* it is evident (by 157) that $2,\dot{6} = 2,\frac{6}{9}$, and that $2,7\dot{5}\dot{3} = 2,7\frac{53}{99}$, and $7\dot{2}5,\dot{6} = 7\frac{25,6}{99,9}$ and ſo on; but $2,\frac{6}{9} = \frac{24}{9}$, and this multiplied by 9, becomes barely 24; but any Number multiplied by 9, is the ſame as when multiplied by 10, and once ſubtracted; whence $\overline{2,\dot{6} \times 10,}— 2,\dot{6} = 24$. Hence this *general Rule* is deduced.

From the mixed Repetend multiplied by 10, ſubtract itſelf; or (which is the ſame Thing) from any mixed Repetend ſubtract the terminate Part, and to the Logarithm of the Remainder, add the Arithmetical Complement of as many Nines as there are Places of Figures in the Repetend, and the Sum will be the Logarithm of the ſaid mixed Repetend.

Example 1. Required the Logarithm of $2,\dot{6}$?

From the given Repetend ——	$2,\dot{6}$	
Subtract the terminate Part ——	2	
To the Logarithm of the Remainder	2,4 =	0,3802112
Add the Arithmetical Complement of	9 =	0,0457575
The Sum is the Logarithm of ——	$2,\dot{6}$ =	0,4259687.

Example 2. Required the Logarithm of $2,7\dot{5}\dot{3}$?

From the given Repetend ——	$2,7\dot{5}\dot{3}$	
Subtract the terminate Part ——	27	
Then to the Logarithm of the Remainder	2,726 =	0,4355258
Add the Arithmetical Complement of the Logarithm of —— ——	99 =	0,0043648
The Sum is the Logarithm of the Repetend —— —— ——	$2,7\dot{5}\dot{3}$ =	0,4398906.

Example 3. Required the Logarithm of $7\dot{2}5,\dot{6}$?

From the Repetend —— ——	$7\dot{2}5,\dot{6}$	
Subduct the conſtant Part ——	7	
To the Logarithm of the Remainder	724,9 =	2,8602781
Add the Arithmetical Compliment of —	999 =	0,0004345
The Sum is the Logarithm of ——	$7\dot{2}5,\dot{6}$ =	2,8607126.

161. The Learner is now prepared to work any of the common Rules of Arithmetic, where the Operations are tedious or

difficult, in a very eaſy and conciſe Manner, by *Logarithms*, of which take the following Examples.

Example 1. In Duodecimal Multiplication.

F. In. F. In.
Required the Area of 9 : 10 by 8 : 8?

F. In.
Add { The Logarithm of 9 : 10 = 9,83̸ = 0,9927008
To the Logarithm of 8 : 8 = 8,6̸ = 0,9378521

F. In.
Sum is the Logarithm of the Area required. } 85 : 2¾ = 85,2̸ = 1,9305529.

As *Diviſion* is only the Reverſe, it needs no Example.

162. Example 2. In the *Rule of Three Direct.*

If 2 *C.* 3 *Qrs.* 21 *lb.* of Sugar —— 2,9375 = 0,4679778
Coſt 6*l.* 1*s.* 8*d.* —— —— 6,083̸ = 0,7841316
What will 12 *C.* 2 *Qrs.* coſt? —— 12,5 = 1,0969100
1,8810416
Anſwer, 25*l.* 17*s.* 8*d*¾. —— 25,8864 = 1,4130638.

Note, here the Logarithm of the firſt Number is ſubtracted from the Sum of both the others; but a more conciſe Way is to take the *Arithmetical Complement* of the firſt (which we denote by *A. C.*) and then the Whole is performed by *one Addition.*

Example 3. At one Operation.

If ½ *C.* of Tobacco (*A. C.*) — 0,5 = 0,3010300
Coſt 4*l.* 17*s.* 8*d.* —— —— 4,63̸ = 0,6658935
What will 7 *lb.* coſt? —— —— 0,0625 = ,8,7958800
Anſwer, 11*s.* 7*d.* —— —— 0,57916̸ ≐ ,9,7628035.

163. Example 4. *Inverſe Proportion,* at one Operation.

If Wheat be 6*s.* 4*d. per* Buſhel —— 6,3̸ = 0,8016325
And the Penny White Loaf weigh 7¾ *oz.* 7,75 = 0,8893017
What is the Weight, at 3*s.* 10*d. p.* Buſhel (*A. C.*) —— —— —— —— } 3,83̸ = 9,4164234
Anſwer, 12 *Oz.* 16 *Pwt.* 2 *Grs.* — 12,8043 = 1,1073576.

Here we take the Arithmetical Complement (*A. C.*) of the third Term, the Reaſon of which is evident from (156).

164. EXTRACTION of ROOTS by *Logarithms*.

Example 1. Required the *Square Root* of 2830,24?

The Logarithm thereof is —— 3,4518232

One Half is the Logarithm of the Root 53,2 = 1,7259116

Example 2. What is the *Square Root* of 14,6?

The Logarithm of that Number is —— 1,1663314

Half of which is the Logarithm of the Root 3,8297 = 0,5831657

165. Example 3. Required the Cube Root of 1,728?

The Logarithm of —— — 1,728 = 0,2375437

A third of which is the Logarithm of the Root 1,2 = 0,0791812.

Example 4. What is the Cube Root of 0,27580?

The Logarithm thereof is —— —— ,9,4407132

⅓ Part thereof is the Logarithm of — 0,6509 = ,9,8135710.
The Cube Root required.

N. B. In Decimal Numbers, where the Indices are Decimal, in extracting the *Square Root* you add 10 to the Index; for the *Cube Root* 20; for the *Biquadrate Root* 30; and ſo on.

Example 5. Required the ſeveral Roots of the laſt Number, for Inſtance.

The Logarithm of the Number 0,27580 =,9.4407132
The Half is the Logarithm of the *Square* Root —— } 0,52523 =,9.7023566
A third Part, of the *Cube* Root — 0,6509 & =,9.8135710
A Fourth, the *Biquadrate* —— 0,7247 & =,9.8601783
A Fifth, the *Surſolid* Root —— 0,7729 & =,9.8881426

Thus you ſee how extremely eaſy it is to extract any Root out of any Number by Logarithms, and eſpecially in ſuch Caſes where by the common Rules the Operation is very laborious and difficult, and ſometimes quite impracticable.

The Reaſon of extracting Roots by Logarithms, is evident from what we have obſerved of theſe Numbers in (135). Since to Square or Cube any Number, is only to multiply it by it's ſelf once, or twice, we thought it needleſs to give any Examples in Logarithms of that Affair; eſpecially, ſince the are only the Reverſe of the foregoing, and evident from (134.)

166. To find a *Mean Proportional* between any two Numbers, as 3, and 243.

Add the Logarithm of the Numbers	3,=	0.4771212
	243,=	2.3856063
The Sum is	=	2,8627275
Half of which is the Logarithm of the Mean required, *viz.*	27,=	1,4313637

For 3 : 27 : : 27 : 243.

167. To find two Mean Proportionals between any two Numbers, as 27, and 729.

To double the Logarithm of the firſt Number	27,=	1,4313638
		1,4313638
Add the Logarithm of the laſt	729,=	2,8627275
The Sum is		5,7254551
A Third of which is the Logarithm of the *firſt Mean*, viz.	81,=	1,9084850
To this add the Logarithm of the laſt Number again, *viz.*	729,=	2,8627275
The Sum is		4,7712125
Half that is the Logarithm of the ſecond Mean required.	243,=	2,3856062

For 27 : 81 : : 243 : 729. Or thus, having found the *firſ* *Mean*, as here 81, you know then the *Ratio* of the Series 27) 81 (= 3 the Ratio, therefore *from the Logarithm of the firſ* *Mean, ſubtract the Logarithm of the firſt given Number, the Remainder is the Logarithm of the Ratio; which add to the Logarithm of the firſt Mean, the Sum is the Logarithm of the ſecond Mean; and again, added to the Logarithm of the Second, it gives the Logarithm of the third Mean*, and ſo on. All which is evident from what we firſt premiſed concerning the Nature of a Geometrical Series, and Logarithms adapted thereto. See (130, 131, &c.)

We might now have proceeded to the Computations of *Intereſt*, *Annuities*, &c., which are beſt performed by Logarithms, but as the *Theorems* or *Canons* for this Purpoſe muſt be raiſed by an *Algebraic Proceſs*, we muſt defer this Buſineſs till we treat of that Science in the next enſuing Part. Alſo the Uſe of Logarithms in *Navigation*, and other Mathematical Arts, will be largely ſhewn when we come to treat of thoſe Subjects. What we have done at preſent being ſufficient for all the common Parts of Arithmetic, where *Numerical Calculations* only are concerned.

And as to the *Natural* or *Hyperbolical Logarithms* we ſhall refer them to the Doctrine of Fluxions, where we ſhall ſhew how they are made, and how theſe we here treat of are derived from them: Alſo *Logiſtical Logarithms* will be fully treated of in the Practical Part of Aſtronomy, where they are uſed; and *Tables* of every Sort of Logarithms will be ſupplied in their proper Places.

INSTITUTIONS

OF

ALGEBRA.

CHAP. I.

168. ALGEBRA is a Kind of *ſpecious Arithmetic*, as we here make Uſe of *Species*, i. e. Symbols or Letters to repreſent Quantities of every Kind, as well known as unknown. And it is cuſtomary to repreſent *known Quantities* by the firſt Letters of the Alphabet a, b, c, d, *&c.* and *unknown* ones by the laſt Letters, x, y, z.

169. The Terms of an Algebraic Expreſſion are connected under all their various Relations by proper Symbols or Characters, as mentioned in (22.) but we are here to conſider the two principal Symbols + and — in a more extenſive View than we did there, for they are deſigned to repreſent any two contrary Modes, Qualities, or Actions, *&c.* In ſhort, what ever is repreſented by the *affirmative* Sign +, as $+a$; the Contrary is repreſented by the *negative* Sign —, as $-a$.

Thus if $+a$ signifies any Thing	added above,	$-a$ signifies	ſubtracted below,
	to the Right,		to the Left,
	forwards,		backwards.

If $+a$ signifies any	Increaſe,	$-a$ signifies any	Decreaſe,
	Gravity,		Levity,
	Money due,		Money owing,
	Motion upwards,		Motion downward

And ſo on in every Kind of Contrariety. And two ſuch Quantities connected together in any Caſe deſtroy each other's Effect, or are equal to Nothing, as $+a-a=o$. Thus, if a Man has but 10*l.* and at the ſame time owes 10*l.* he is worth *Nothing.*

170. If the firſt Quantity or Term be affirmative we neglect the Sign $+$, as $a+b$ inſtead of $+a+b$. If an Algebraical Quantity conſiſt of two Terms, it is called a *Binomial*, as $a+b$; if of three Terms, a *Trinomial*, as $a+b+c$. If there be more Terms, it is called a *Multinomial.* All which are *compound* Quantities. *Simple* Quantities conſiſting of one Term only, as $+a$, $+ab$, $+abc$.

171. When ſimple Quantities are to be multiplied together we do not, generally, uſe the Symbol $\times$, but place them together without; as ab inſtead of $a\times b$, and abc for $a\times b\times c$; when compound Quantities are to be repreſented as multiplied, then a Line is drawn over the Factors connected with the proper Sign $\times$; as $\overline{a+b}\times c$, $\overline{a+x}\times\overline{b-y}$.

172. When any Number of Figures are prefixed to the Terms of an Algebraic Quantity, they are called *Coefficients*, and ſhew how often the Quantity is to be taken; as $3a$, $5bc$, $2a+5b$, *&c.* and when no Number is prefixed, *Unity* is always underſtood, tho' not expreſſed, for the *Coefficient*; thus a is the ſame as $1a$, and bc as $1bc$, ſince any Thing multiplied by *Unity* is ſtill the ſame.

173. Quantities are ſaid to be *like*, or *ſimilar*, that are repreſented by the ſame Letter or Letters equally repeated; thus $+3a$ and $-5a$ are *like*; but a and b, or a and aa are *unlike*. What other Definitions and Symbols are uſed in *Algebra*, will be explained in their proper Places.

CHAP. II.

ADDITION.

74. IN the ADDITION of Algebraic Quantities there are three CASES, as follow.

CASE I. To add Quantities that are *like*, and have *like Signs*.

RULE.

Add together the Coefficients, to their Sum subjoin the common Quantity, or Letters, and prefix the Sign where neceſſary.

EXAMPLES.

$$\begin{array}{lccc}
\text{To} & 5a & 3a-5b & a+2b-3c \\
\text{Add} & 3a & a-2b & 3a+b-4c \\ \hline
\text{Sum} & 8a & 4a-7b & 4a+3b-7c
\end{array}$$

$$\begin{array}{ll}
\text{To} & ab-5b+3x-21y+15z+7 \\
\text{Add} & 4ab-b+7x-4y+9z+10 \\ \hline
\text{Sum} & 5ab-6b+10x-25y+24z+17
\end{array}$$

175. CASE II. To add Quantities that are *like*, but have *nlike Signs*.

RULE.

Subtract the leſſer Coefficient from the greater, to the Remainder refix the Sign of the greater, and ſubjoin the common Letter or Quantities.

EXAMPLES.

$$\begin{array}{lcc}
\text{To} & -4a & -5b-6c-9x \\
\text{Add} & +7a & +3b+8c+5x \\ \hline
\text{Sum} & +3a & -2b+2c-4x
\end{array}$$

$$\begin{array}{lcc}
\text{To} & a+6x-5y+8 & 2a-2b+13 \\
\text{Add} & -5a-4x+4y-3 & -2a+2b-10 \\ \hline
\text{Sum} & -4a+2x-y+5 & 0\quad 0\quad +3
\end{array}$$

We proceed here according to Cuſtom, but it is with ſome mpropriety that we talk of *adding* Quantities with unlike Signs,

ſince the Operation does wholly conſiſt in *Subtraction*, as it mu
from the Nature of the Signs (169, 173.)

176. CASE III. To add Quantities that are unlike.

RULE.

Set them all down one after another, with their Signs and Coefficients prefixed.

EXAMPLES.

$$\begin{array}{llll} \text{To} & 2a & 3a & 2a - 5bc \\ \text{Add} & 3b & -4x & 2x + 7 \\ \hline \text{Sum} & 2a + 3b & 3a - 4x & 2a - 5bc + 2x + \end{array}$$

$$\begin{array}{ll} \text{To} & 4a + 4b + 3c - 9 \\ \text{Add} & -4x - 4y + 3z \\ \hline \text{Sum} & 4a + 4b + 3c - 4x - 4y + 3z - 9 \end{array}$$

CHAP. III.

SUBTRACTION.

177. SINCE the Sign — is juſt oppoſite to the Sign +, and *Subtraction* juſt contrary to *Addition*, therefore to ſubtract a Quantity is the very ſame Thing as to *add* the ſame Quantity with the contrary Sign. Or thus,

If I am to receive 3 *l.* —— —— $+3a$
or to pay 3 *l.* —— —— —— $-3a$
The Odds or Difference to me, is —— $+6a$

Or, if this Day I have 10 *l.* —— —— $+10a$
And Yeſterday I owed 3 *l.* more than I had to pay, — — } —— $-3a$
The Odds in my Fortune is 13 *l.* better to-Day than Yeſterday, — } —— $+13a$

Whereas it would have been but 10 *l.* better, had not the negative 3 *l.* been ſubducted. Therefore we have a

GENERAL RULE.

178. *Change the Signs of the Quantity to be ſubtracted,* and then add them both together, by the Rules of the preceding Chapter; the Sum ariſing by ſuch Addition is the true Remainder.

EXAMPLES.

From	$+8a$	$4a - 7b$
Take	$+3a$	$3a - 5b$
Remains	$8a - 3a = 5a$	$a - 2b$

From	$3a$	$-2b + 2c - 4x$
Take	$-4a$	$-5b - 6c - 9x$
Remains	$+7a$	$3b + 8c + 5x$

From	3	$2a - 3x + 5y - 6$
Take	$2a - 2b + 13$	$6a + 4x + 5y + 4$
Remains	$-2a + 2b - 10$	$-4a - 7x \quad 0 \quad -10$

CHAP. IV.

MULTIPLICATION.

179. IN Multiplication there is one General Rule for the Signs; viz. *When the Signs of the Factors are like, (that is,* both +, or both —) *the Sign of the Product is* + ; *but when the Signs of the Factors are unlike, the Sign of the Product is* —. This General Rule will reſolve it ſelf into *Four particular Caſes,* which we ſhall illuſtrate ſeparately in ſimple Quantities.

180. CASE I. When any poſitive Quantity as $+a$ is multiplied by a poſitive Number $+n$, the Meaning is, That $+a$ is to be taken ſo many times as there are Units in n; and the Product is evidently n times a, or na.

EXAMPLES.

Multiply	$+a$	$2a$	$5bx$	$9dc$
By	$+n$	$3b$	7	$4y$
Product	na	$6ab$	$35bx$	$36ydc.$

CASE

181. CASE II. When $-a$ is multiplied by n, then $-a$ is to be taken as often as there are Units in n, and the Product muſt be n times $-a$, or $-na$.

EXAMPLES.

Multiply	$-a$	$-2a$	$-5bx$	$-9dc$
By	$+n$	$3b$	7	$4y$
Product	$-na$	$-6ab$	$-35bx$	$-36dcy$.

182. CASE III. As Multiplication by a *poſitive Number* implies a *repeated Addition*, ſo Multiplication by a *Negative* implies a *repeated Subtraction*; and therefore when $+a$ is to be multiplied by $-n$, it means only that $+a$ is to be ſubtracted as often as there are Units in n; and therefore the Product being *negative* muſt alſo be $-na$.

EXAMPLES.

Multiply	$+a$	$2a$	$5bx$	$9dc$
By	$-n$	$3b$	-7	$-4y$
Product	$-na$	$-6ab$	$-35bx$	$-36dcy$.

183. CASE. IV. When $-a$ is to be multiplied by $-n$, then $-a$ is to be ſubtracted as often as there are Units in n; but to ſubtract $-a$ is equivalent to *adding* $+a$, (177.) therefore this Caſe is the ſame in Effect with the Product, and is evidently $+na$.

EXAMPLES.

Multiply	$-a$	$-2a$	$-5bx$	$-9de$
By	$-n$	$-3b$	-7	$-4y$
Product	$+na$	$+6ab$	$+35bx$	$+36dcy$.

184. When the Factors are one or both *compound Quantities*, or conſiſt of ſeveral Parts; you muſt *multiply every Part of the Multiplicand by each Part of the Multiplier; and then add all the Products into one Sum; and that Sum ſhall be the Product required.*

EXAMPLES.

$$\begin{array}{llll}
\text{Mult.} & a+b & a-b+c & a-2y+5z-6 \\
\text{By} & a & -b & 8 \\
\hline
\text{Prod.} & aa+ab & -ab+bb-bc & 8a-16y+40z-48 \\
\hline
\end{array}$$

$$\begin{array}{lll}
\text{Mult.} & 4a+b & 2a-4b \\
\text{By} & a-2b & 2a+4b \\
\hline
 & 4aa+ab & 4aa-8ab \\
 & \quad -8ab-2bb & 8ab-16bb \\
\hline
\text{Product} & 4aa-7ab-2bb & 4aa \quad 0 \quad -16bb \\
\hline
\end{array}$$

$$\begin{array}{lll}
\text{Mult} & xx-ax & aa+ab+bb \\
\text{By} & x+a & a-b \\
\hline
 & xxx-axx & aaa+aab+abb \\
 & \quad +axx-aax & \quad -aab-abb-bbb \\
\hline
\text{Prod.} & xxx \quad 0 \quad -aax & aaa \quad 0 \quad 0 \quad -bbb \\
\hline
\end{array}$$

185. I ſhall here inſert one Example more to ſhew the Reaſon of the common Method of proving the Work of Multiplication in Numbers, *by caſting out the Nines in the Factors, and Product of their Remainders*, and alſo out of the *general Product, to obſerve the Equality of the Remainders.* (See *Inſt.* 55.) We ſhall take the firſt Example in (*Inſt.* 54,) where $1750 \times 76 = 133000$. Therefore

$$\begin{array}{lll}
\text{Mult.} & 9a+c & =1750=\overline{9\times 194}+4. \\
\text{By} & 9b+d & =\quad 76=9\times 8\;+4. \\
\hline
 & 81ab+9bc & \quad 10500 \\
 & \quad +9ad+cd & \quad 12250 \\
\hline
\text{Prod.} & \multicolumn{2}{l}{81ab+9bc+9ad+cd=133000}
\end{array}
\left\{\begin{array}{l} a=194 \\ c=\;\;4 \\ b=\;\;8 \\ d=\;\;4 \end{array}\right.$$

Now here we are to obſerve, that any Number divided by 9 leaves the ſame Remainder as when the Figures of that Number are added together, and the Nines caſt out as often as they occur, thus $76 \div 9$, leaves 4; and $7 + 6 = 9 + 4$. Alſo $1750 \div 9$ leaves 4, and $1 + 7 + 5 + 0 = 9 + 4$: Laſtly, $133000 \div 9$ leaves 7, and $1 + 3 + 3 = 7$. And thus it will be for every other Number.

Again, 'tis evident, that the Sum of the three firſt Terms of the *Algebraic Product* divided by 9 leaves no Remainder, what Remainder therefore is, muſt be from the fourth Term $c\,d$ divided by 9; But this Term is always the Product of the two Remainders of the Factors, *c* and *d*; conſequently, if *the Product of theſe Remainders, divided by Nine, leave the ſame Remainer as the Figures of the Product of the two Factors when added together, and the Nines caſt out, the Work will be right*; provided no Error be committed that amounts to Nine, or any Multiple of Nine. *N.B.* I have inſerted this Demonſtration of the Proceſs here, as it is an *Algebraic One*, and what has been deſired by many Perſons, who have ſought for it in vain in Books of *Arithmetic* hitherto publiſhed.

CHAP. V.

DIVISION.

168. IN DIVISION of *Algebraic Quantities* the Rule for the Signs is the ſame as in *Multiplication*, viz. *If the Signs of the Diviſor and Dividend are like, the Sign of the Quotient muſt be* +; *but if they are unlike, the Sign of the Quotient muſt be* —. This is evidently deduced from the Rule in *Multiplication* (179.) if it be conſidered, *that the Quotient muſt be ſuch a Quantity as multiplied by the Diviſor, ſhall give the Dividend.*

And this is a Gereral Rule for all Operations in *Diviſion*, which are only the Reverſe of *Multiplication*, and will be eaſy to underſtand when illuſtrated by Examples, as follow.

$na\,(\,n; \qquad -a\,)-na\,(+n; \qquad -a\,)+na\,(-n.$

$2a\,)\,6ab\,(\,3b; \qquad a\,)\,aa+ab\,(\,a+b$
$6ab \qquad\qquad\quad aa$
$\overline{\ ..\ } \qquad\qquad\quad \overline{..}\ ab$
$\qquad\qquad\qquad\qquad\quad ab$
$\qquad\qquad\qquad\qquad\quad \overline{..}$

$4a+b\,)\,4aa-7ab-2bb\,(\,a-2b$
$4aa+\ ab$
$\overline{-8ab-2bb}$
$-8ab-2bb$
$\overline{\ \ ..\qquad ..\ }$

$2a-4b\,)\,4aa-16bb\,(\,2a+4b$
$4aa-\ 8ab$
$\overline{+8ab-16bb}$
$+8ab-16bb$
$\overline{\ \ ..\qquad ..\ }$

$a-b\,)\,aaa-bbb\,(\,aa+ab+bb$
$aaa-aab$
$\overline{+aab-bbb}$
$+aab-abb$
$\overline{+abb-bbb}$
$+abb-bbb$
$\overline{\ \ ..\qquad ..\ }$

$$\begin{array}{l}
3a - 6\ )\ 6aaaa - 96\ (\ 2aaa + 4aa + 8a + 16 \\
\qquad\quad\ \underline{6aaaa - 12aaa} \\
\qquad\qquad\qquad 12aaa - 96 \\
\qquad\qquad\qquad \underline{12aaa - 24aa} \\
\qquad\qquad\qquad\qquad\quad 24aa - 96 \\
\qquad\qquad\qquad\qquad\quad \underline{24aa - 48a} \\
\qquad\qquad\qquad\qquad\qquad\quad 48a - 96 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{48a - 96} \\
\qquad\qquad\qquad\qquad\qquad\quad\ \ ..\qquad ..
\end{array}$$

187. In dividing, when you come to a Remainder of Term, it is commonly ſet down with the Diviſor under it, a the other Terms, which together make the whole Quotie Thus

$$\begin{array}{l}
a + x\ )\ aa + xx\ (\ a - x + \dfrac{2xx}{a+x} \\
\qquad\quad\ \underline{aa + ax} \\
\qquad\quad\ - ax + xx \\
\qquad\quad\ \underline{- ax - xx} \\
\qquad\qquad\qquad 2xx
\end{array}$$

188. It ſometimes happens, that the ſame Letter or Quant is found in all the Terms of the Diviſor and Dividend, and th there is obviouſly ſome common Meaſure to the Coefficients the Terms ; when this is the Caſe, you expunge the comm Quantity, divide by the common Meaſure, and place the Divi ſo reduced, under the new Dividend in the Quotient. Th

$2b\ )\ ab + bb\ \left(\dfrac{a+b}{2}\right.$, here Unity, or 1 is the common Me ſure. Again,

$$20ad\ )\ 10ab + 15ac\ \left(= \frac{2b + 3c}{4d}\right.$$

Thus $12ab\ )\ 30ax - 54ay\ \left(= \dfrac{5x - 9y}{2b}\right.$

And $4aa\ )\ 8ab + 6ac\ \left(= \dfrac{4b + 3c}{2a}\right.$

CHAP. VI.

Of *FRACTIONS.*

9. *ALgebraic Fractions* are of the ſame Nature, and require the ſame Management as thoſe of Numbers, for ppoſe $a = 2$, and $b = 3$; then $\frac{2}{3} = \frac{a}{b}$, a *proper Fraction*; $\frac{3}{2} = \frac{b}{a}$, an *improper Fraction*: and $2\frac{2}{3} = a + \frac{a}{b}$ a ixed fractional Quantity.

190. A *mixed Quantity* is reduced to an improper Fraction by e Rule in (Inſt. 68.) *viz. Multiply the integral Part by the Deminator of the fractional Part, to which Product add the Numerar; and the Sum will be a new Numerator, under which write the enominator, and it is the improper Fraction required.* Thus $+ \frac{a}{b}$ becomes $\frac{ab + a}{b}$; $a + b + \frac{x}{z}$ is $\frac{az + bz + x}{z}$; and $- x + \frac{aa - ax}{x} = \frac{ax - xx + aa - ax}{x} = \frac{aa - xx}{x}$; ſo $- x + \frac{2xx}{a + x} = \frac{aa + xx}{a + x}$.

191. Fractions of different Denominations are reduced to the me Denomination thus; *Multiply all the Denominators together r a common Denominator, and each Numerator by every Deninator but its own, for a new Numerator.* (See Inſt. 69.) So $\frac{b}{c}$, $\frac{c}{d}$ will become $\frac{acd}{bcd}$, $\frac{bbd}{bcd}$, $\frac{ccb}{bcd}$. Thus $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ $= \frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{edb}{bdf} = \frac{adf + cbf + edb}{bdf}$.

192. A Fraction is reduced to its loweſt Terms by the Rule in Inſt. 66.) for finding a *common Diviſor*; thus the Fraction $\frac{5ab}{1bx}$ has its common Meaſure, 3 for the Coefficient, and *b* ſor the ther Part; and ſo the Whole is $3b$, by which dividing the Fracion, it is reduced to its loweſt Terms of the ſame Value, *viz.*

$\frac{5a}{27x} = \frac{15ab}{81bx}$. Thus $\frac{bbc+bbd}{cd+dd} = \frac{b^2}{d}$; and $\frac{aaa+bbb}{aa-bb} =$ $\frac{aa-ab+bb}{a-b}$, being divided by $a+b$ the common Meaſure of the Numerator and Denominator. *N. B.* It often happens that the ſame Letter or Letters are contained in every Part or Term of the Fraction, which in ſuch a Caſe are to be expunged, and the Fraction is thereby reduced to a more ſimple Form of the ſame Value. Thus $\frac{ab+db}{bx} = \frac{a+d}{x}$; and $\frac{25za}{5xz+15az}$ $= \frac{5a}{x+3a}$ by expunging z, and dividing by 5.

193. When Fractions are to be *added*, *ſubtracted*, *multiplied*, or *divided*, they ſhould be firſt reduced to one Denomination, and in their loweſt Terms, and then the Rules for the Operations are the ſame as for *numeral Fractions*. Thus to add $\frac{a}{b}$ to $\frac{c}{d}$, you reduce them to a common Denomination $\frac{ad}{bd}$, and $\frac{bc}{bd}$, then their Sum is $\frac{ad+bc}{bd}$ and $\frac{a}{b} + \frac{c}{d} + \frac{d}{e} = \frac{ade+bce+ddb}{bde}$.

194. To *ſubtract* one Fraction from another, you reduce them to a common Denominator, and then take the Difference of the Numerators, and under-write the common Denominator. Thus if from $\frac{a}{b}$ you take $\frac{c}{d}$, the Difference is $\frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$; from the Integer a take the Fraction $\frac{b}{c}$, the Difference is $a - \frac{b}{c}$, from $\frac{a+x}{b}$ take $\frac{a-x}{c}$, their remains $\frac{ac+cx}{bc} - \frac{ab-bx}{bc}$, that is, $\frac{ac+cx-ab+bx}{bc}$.

195. To *multiply* one Fraction by another, *you multiply the Numerators one into the other for the Numerator of the Product, and the Denominators multiplied, one into another, give the Denominator of the Product.* Thus $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; and $\frac{a+b}{c} \times \frac{a-b}{d} =$ $\frac{aa-bb}{cd}$.

$\frac{aa-bb}{cd}$. Alſo $\overline{a+\frac{b}{c}}\times\frac{d}{e}=\frac{\overline{ac+b}}{c}\times\frac{d}{e}=\frac{acd+bd}{ce}$. $a\times\frac{b}{c}=\frac{a}{1}\times\frac{b}{c}=\frac{ab}{c}$; for any Integer, a, is reduced to the Form of a Fraction by writing Unity under it, as $\frac{a}{1}$.

196. To divide one Fraction by another, *multiply the Numerator of the Dividend by the Denominator of the Diviſor, the Product will be the Numerator of the Quotient. Then the Denominator of the Dividend, multiplied by the Numerator of the Diviſor, gives the Denominator of the Quotient.* Thus $\frac{a}{b}\Big)\,\frac{c}{d}\,\Big(=\frac{cb}{ad}$; and $\frac{d}{e}\Big)\,\frac{acd+bd}{ce}\Big(=\frac{acde+bde}{cde}=\frac{ac+b}{c}=a+\frac{b}{c}$. And $\frac{a-b}{d}\Big)\,\frac{aa-bb}{cd}\,\Big(=\frac{a+b}{c}$, after Reduction; and $\frac{a+b}{a-b}\Big)\,\frac{a-b}{a}$ $\Big(=\frac{aa-2ab+bb}{aa+ab}$.

197. In order to demonſtrate the Truth, or ſhew the Reaſon, of the foregoing Rules for the *Addition*, *Subtraction*, *Multiplication* and *Diviſion* of Fractions, we muſt here premiſe the following *Axioms*, or *Principles* that are *in themſelves evident Truths*; theſe will be alſo neceſſary in moſt of our future Mathematical Speculations.

198. AXIOM I. *Things that are equal to one and the ſame Thing, are equal to one another.*

Thus, if $a=m$, and $b=m$, then $a=b$.

199. AXIOM II. *If to equal Things, you add equal Things, the Sums will be equal.*

Thus, if to the equal Quantities . . . . $a=m$
You add the equal Quantities . . . . . $\underline{b=n}$
The Sums will be equal . . . . $a+b=m+n$.

200. AXIOM III. *If from equal Things* . . $a=m$
You ſubduct equal Things . . . . $\underline{b=n}$
The Remainders will be equal, viz. $a-b=m-n$.

201. AXIOM IV. *If equal Things* . . . $a = m$
Be multiplied by equal Things . . $\underline{b = n}$
The Product will be equal . . . . $ab = mn.$

202. AXIOM V. *If equal Things* . . . . $a = m$
Be divided by equal Things . . . . $\underline{b = n}$
The Quotients will be equal, viz. $\frac{a}{b} = \frac{m}{n}.$

203. Now from hence we prove that Fractions of any Kind reduced to the ſame Denomination are *added*, by *adding their Numerators, and ſubſcribing the common Denominator*. Thus $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ for let $\frac{a}{b} = m$, $\frac{c}{b} = n$, and multiplying both Sides of each Equation by b, we have $a = bm$, and $c = bn$ (by Inſt. 201) and $mb + bn = a + c$, (Inſt. 199); and $m + n = \frac{a+c}{b}$ (Inſt. 202.); that is, (by ſubſtituting the Value of m and n) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. And in the ſame Manner it is ſhewn, that $\frac{a}{b} - \frac{c}{b} = m - n = \frac{a-c}{b}$.

204. Again, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ is thus demonſtrated. Let $\frac{a}{b} = m$, $\frac{c}{d} = n$; then $a = mb$, and $c = nd$; and $bdmn = ac$ (Inſt. 201.) therefore $\frac{ac}{bd} = m \times n = \frac{a}{b} \times \frac{c}{d}$.

205. Laſtly, it is ſhewn that $\frac{a}{b}$ divided by $\frac{c}{d}$ give $\frac{ad}{cb}$; for $mb = a$, and $mbd = ad$ (Inſt. 201); alſo $nd = c$, and $ndb = cb$; therefore $\frac{mbd}{ndb} = \frac{ad}{cb} = \frac{m}{n} = \frac{a}{b} \div \frac{c}{d}$.

CHAP.

CHAP. VII.

Of INFINITE SERIES.

206. WHEN it happens in Diviſion, that the Diviſor is not exactly contained in the Dividend, the Operation may be continued without End; and the Quotient will in that Caſe be an *Infinite Series* of Terms. This will be the Caſe, if you divide *Unity* by $1-a$; as below.

$$
\begin{array}{l}
1-a\,)\,1\,(\,1+a+aa+aaa,\ \&c.\\
\quad \underline{1-a}\\
\quad\quad +a\\
\quad\quad \underline{+a-aa}\\
\quad\quad\quad +aa\\
\quad\quad\quad \underline{+aa-aaa}\\
\quad\quad\quad\quad +aaa\\
\quad\quad\quad\quad \underline{+aaa-aaaa}\\
\quad\quad\quad\quad\quad +aaaa,\ \&c.
\end{array}
$$

N. B. Here it is ſoon to be obſerved, in what Order or Manner the ſeveral Terms of the Series in the Quotient will ariſe, without farther Operation; this is called *diſcovering the Law of the Series.*

207. Let it be required to divide $aa+xx$ by $a+x$; thus

$$
\begin{array}{l}
a+x\,)\,aa+xx\,(\,a-x+\dfrac{2xx}{a}-\dfrac{2x^3}{a^2}+\dfrac{2x^4}{a^3},\ \&c.\\
\quad \underline{aa+ax}\\
\quad -ax+xx\\
\quad \underline{-ax-xx}\\
\quad\quad +2xx\\
\quad\quad \underline{+2xx+\dfrac{2xxx}{a}}\\
\quad\quad\quad -\dfrac{2xxx}{a}\\
\quad\quad\quad \underline{-\dfrac{2xxx}{a}-\dfrac{2xxxx}{a^2}}\\
\quad\quad\quad\quad +\dfrac{2xxxx}{a^2},\ \&c.
\end{array}
$$

Here the Law of the Series is diſcovered in a few Terms; and the Series may be continued at Pleaſure without farther Trouble; alſo the Signs are here alternately + and —.

208. Another Example take as follows. Divide ay by $1 + x$.

$$\begin{array}{llll}
1+x\,)\,ay & \multicolumn{3}{l}{(\,ay-ayx+ayxx-ayxxx,\ \text{\&c.}} \\
\quad\underline{ay+ayx} & & & \\
\quad\quad -ayx & & & \\
\quad\quad \underline{-ayx-ayxx} & & & \\
\quad\quad\quad +ayxx & & & \\
\quad\quad\quad \underline{+ayxx+ayxxx} & & & \\
\quad\quad\quad\quad -ayxxx-ayx^4 & & & \\
\quad\quad\quad\quad \underline{-ayxxx-ayx^4} & & & \\
\quad\quad\quad\quad\quad +ayx^4,\ \text{\&c.} & & &
\end{array}$$

Therefore $\frac{ay}{1+x} = ay - ayx + ayxx - ayxxx$, &c. $= ay \times \overline{1 - x + xx - xxx}$, &c. but alſo $\frac{ay}{1+x} = ay \times \frac{1}{1+x}$; hence $\frac{1}{1+x} = 1 - x + xx - xxx$, &c. as is evident by dividing each Side of the Equation by ay.

209. This Method of expreſſing a Fraction in an *Infinite Series* will be often found very advantageous in approximating the Values of Mathematical Quanties expreſſed in an Algebraic or Fluxionary Manner, as we ſhall find in many Inſtances as we proceed. And indeed this is the Foundation of the *Arithmetic of Infinities*; for theſe interminate Series may be *added* and *ſubtracted*; *multiplied* and *divided*; *ſquared* and *cubed*; and the *Square* and *Cube Roots* extracted; and ſo that the *Sum* and *Difference*; *Product* and *Quotient*; *Power* and *Root*, ſhall ſtill be an *infinite Series*, of which we ſhall treat more fully hereafter, when the Learner has ſeen more of the Nature and Uſe of this Sort of Arithmetic.

CHAP. VIII.

The RULES *for ordering ſimple* EQUATIONS.

210. AN EQUATION *is the Expreſſion of Equality between two Quantities*; as $\frac{3}{4} x = b$, or $3 x = 4 b$, or $x = \frac{4}{3} b$. Now the Uſe of an Equation is to give the Value of an unknown Quantity x on one Side, in others which are known on the other Side; for though $\frac{3}{4} x = b = 9$, it will not ſo eaſily appear what x is, till you get it by it's ſelf on one Side of the Equation, by reaſoning thus; if $\frac{3}{4} x = 9$, then $3 x = 4 \times 9 = 36$, (by Axiom 4.) and ſo $x = \frac{36}{3} = 12$, (by Axiom 5.) and is therefore known.

211. And it is the whole Deſign of this Analytic Science to expreſs the Parts and Conditions of any Problem or Queſtion in Symbols, and to ſupply Rules for the due ordering and forming ſuch a Proceſs as ſhall at laſt produce an Equation, with the unknown Quantity on one Side, and thoſe that are known on the other. The Rules for this Purpoſe are as follow.

RULE I.

212. *Any Quantity may be tranſpoſed from one Side of an Equation to the other, by changing its Sign.* For this is nothing more than to add the ſame Quantity on both Sides with a different Sign. (See Inſt. 199.)

Thus, ſuppoſe $x + 8 = 53$.
Then by Tranſpoſition, $x = 53 - 8 = 45$.
Again, let $5 x - 4 b = 4 x + 10$
By Tranſpoſition, $5 x - 4 x = x = 10 + 4 b$.
If $2 x + a = x + b$.
Then $2 x - x = x = b - a$.

RULE II.

213. *Any Quantity by which the unknown Quantity is multiplied may be taken away, by dividing all the Quantities in the Equation by* t. This is evident from Inſt. 202.

Thus, if $a x = b$, then dividing both Sides by a, we have $= \frac{b}{a}$.

Again, ſuppoſe $3x + 12 = 27$

Then, by Rule I. $3x = 27 - 12 = 15$

And, by Rule II. $x = \frac{15}{3} = 5$.

Alſo, if $ax + 2ba = 3c$

Then, Rule I. $ax = 3c - 2ba$

And by Rule II. $x = \frac{3c}{a} - 2b$.

RULE III.

214. *If the unknown Quantity be divided by any Quantity, that Quantity may be taken away by multiplying all the other Parts of the Equation by it.* (See Inſt. 201.)

Thus, if $\frac{x}{b} = b + 5$

Then ſhall $x = bb + 5b$.

If $\frac{x}{5} + 4 = 10$

Then $x + 20 = 50$

And ſo $x = 50 - 20 = 30$, by Rule 1.

If $\frac{4x}{3} + 24 = 2x + 6$

Then $4x + 72 = 6x + 18$

And (Rule 1.) $72 - 18 = 6x - 4x = 2x = 54$

Therefore $x = \frac{54}{2} = 27$.

RULE IV.

215. *If the unknown Quantity be concerned in Fractions, and there be more ſuch Fractions than one, they may be reduced to a common Denominator, by which, if you multiply all the Terms, the unknown Quantity will be diſengaged as before.*

Thus, let $\frac{x}{5} + \frac{x}{3} = x - 7$

Then $\frac{3x + 5x}{15} = \frac{8x}{15} = x - 7$

Conſequently, $8x = 15x - 105$

Whence, $7x = 105$, and $x = \frac{105}{7} = 15$.

RULE V.

216. *If the unknown Quantity be contained in a Surd Root, it will be equated with the known Quantities by involving both Sides of the Equation, to the proper Power.*

Thus, ſuppoſe . . . . $\sqrt{4x + 16} = 12$

Then, by ſquaring each Side, $4x + 16 = 144$

By Tranſpoſition . . . . $4x = 144 - 16 = 128$

Therefore, ... $x = \frac{128}{4} = 32.$

Again, if . . . $\sqrt{ax + b^2} - c = d$

Then, ... $\sqrt{ax + b^2} = d + c$

And, by ſquaring $ax + b^2 = d^2 + 2dc + c^2$

Whence . . . . . . $x = \frac{d^2 + 2dc + c^2 - b^2}{a}.$

Laſtly, if . . . . $\sqrt[3]{a^2 x - b^2 x} = a$

Then, ... $a^2 x - b^2 x = a^3$

And, ... $x = \frac{a^3}{a^2 - b^2}.$

RULE VI.

217. *If that Side of the Equation which contains the unknown Quantity be a compleat Square, Cube, or other Power; then will the unknown Quantity be equated with known Ones by Extraction of the proper Root.*

For Example, let $x^2 = 144$

Then, ... $x = \sqrt{144} = 12.$

Again, let $x^2 + 6x + 9 = 20$

Then, . . $x + 3 = \sqrt{20}$

And, . . . $x = \sqrt{20} - 3.$

If we have $x^2 + ax + \frac{a^2}{4} - b^2$

Then,

Then, $x+\frac{a}{2}=\pm b^2$*

And, $x=\pm b^2-\frac{a}{2}$.

RULE VII.

218. *If the unknown Quantity be contained in the Terms of an Analogy, it may be had by multiplying Extremes and Means together for an Equation.* (See Inſt. 104, 105.)

Thus, ſuppoſe $12-x:\frac{x}{2}::4:1$

Then, ... $12-x=2x$, and $12=3x$, and $x=4$.

Or, if $20-x:x::7:3$
Then, $60-3x=7x$; or $10x=60$, and $x=6$.

RULE VIII.

219. *If any Quantity be found on both Sides the Equation, or multiplied into all the Terms, or dividing them all, it may be ſtruck out of the Equation.* Thus,

If $3x+b=a+b$; then $3x=a$, and $x=\frac{a}{3}$.

Again, if $3ax+5ab=8ac$; then $3x+5b=8c$, and $x=\frac{8c-5b}{3}$.

If $\frac{2x}{3}+\frac{8}{3}=\frac{16}{3}$; then $2x+8=16$, and $x=4$.

RULE IX.

220. *Inſtead of any Quantity in an Equation you may ſubſtitute another of equal Value.*

Thus, if $3x+y=24$; and $y=9$;

Then, $3x+9=24$; and $x=\frac{24-9}{3}=5$.

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* The Reaſon why both the Signs + and − are here placed before b^2 will be ſhewn a little farther on.

The Rules hitherto delivered relate to Equations which contain but one unknown Quantity; but when one or more are concerned in the Queſtion, you proceed by the following Rules.

RULE X.

221. *If there be two unknown Quantities* x *and* y, *there muſt be two Equations ariſing from the Conditions of the Queſtion; from which a Value of either* x *or* y *muſt be found in each Equation, and putting theſe Values equal to each other, a new Equation will ariſe, involving only one unknown Quantity, which is then found by the foregoing Rules.*

EXAMPLE I.

Suppoſe the Sum of any two Quantities $x + y = s$

And their Difference $x - y = d$

Then we have . . $\begin{cases} x = s - y \\ x = d + y \end{cases}$

Therefore . . . $d + y = s - y$

Conſequently . . . $2y = s - d$

Hence . . . . $y = \frac{s - d}{2}$

And by adding the two Equations $x = \frac{s + d}{2}$.

EXAMPLE II.

Suppoſe $x + y = s$

And let $x : y :: a : b$

Then . . . $bx = ay$

And . . . . $x = \frac{ay}{b}$

But alſo . . . $x = s - y$

Therefore . . . $s - y = \frac{ay}{b}$

Hence . . . . $sb - by = ay$

And . . . . . $sb = ay + by$

Conſequently $y = \frac{sb}{a + b}$

And, laſtly $x = \frac{ay}{b} = \frac{sa}{a+b}$.

EXAMPLE III.

Let $x + y = s$

And $x^2 - y^2 = d$

Then $x = s - y$

The Square of which, $x^2 = s^2 - 2sy + y^2$

Alſo $x^2 = d + y^2$

Therefore $d + y^2 = s^2 - 2sy + y^2$

Hence $d = s^2 - 2sy$

And $2sy = s^2 - d$

So $y = \frac{s^2 - d}{2s}$

And $x = \frac{s^2 + d}{2s}$.

RULE XI.

222. *When there are three unknown Quantities* x, y, *and* z, *there muſt be three Equations given, by which they may be determined; by comparing and equating theſe, two Equations may be obtained involving only two unknown Quantities, which are then known by the foregoing Rule.*

EXAMPLE.

Suppoſe $\begin{cases} x + y + z = 12 \\ x + 2y + 3z = 20 \\ \frac{x}{3} + \frac{y}{2} + z = 6 \end{cases}$

Then we have $\begin{cases} 1.\ x = 12 - y - z. \\ 2.\ x = 20 - 2y - 3z. \\ 3.\ x = 18 - \frac{3}{2}y - 3z. \end{cases}$

From whence we have theſe two Equations $\begin{cases} 12 - y - z = 20 - 2y - 3z \\ 12 - y - z = 18 - \frac{3}{2}y - 3z. \end{cases}$

Therefore we have (by Rule X.) $y = 4$, $z = 2$, and conſequently $x = 6$.

223. Sometimes the Equations are ſuch, that the ſame Quantities in different Equations may have contrary Signs, and deſtroy each other; or be otherwiſe affected, ſo as to ſhorten the common Proceſs very much.

Thus, ſuppoſe $\begin{cases} x + y + z = 26 \\ x - y = 4 \\ x - z = 6 \end{cases}$

Then by Addition only, $3x = 36$

Hence $x = 12$; $y = x - 4 = 8$; and $z = x - 6 = 6$.

RULE XII.

224. *If, in a general Way, the unknown Quantities x and y have Coefficients, and the Value of the Equations are expreſſed in Symbols, as thus,* $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ *it will always be* $y = \dfrac{af - dc}{ae - db}$, *and* $x = \dfrac{ce - bf}{ae - db}$; *which are general Theorems for the Values of x and y in all Caſes.*

For in the firſt Equation, $ax = c - by$, and $x = \dfrac{c - by}{a}$.

And in the Second, $dx = f - ey$, and $x = \dfrac{f - ey}{d}$.

Therefore $\dfrac{c - by}{a} = \dfrac{f - ey}{d}$, and $cd - dby = af - aey$.

Whence $aey - dby = af - cd$.

Conſequently $y = \dfrac{af - dc}{ae - db}$; and $x = \dfrac{ce - bf}{ae - db}$.

Example in Numbers. Suppoſe $\begin{cases} 5x + 7y = 100 \\ 3x + 8y = 80 \end{cases}$

Then $y = \dfrac{5 \times 80 - 3 \times 100}{5 \times 8 - 3 \times 7} = \dfrac{100}{19} = 5\dfrac{5}{19}$, and $x = \dfrac{240}{19} = 12\dfrac{12}{19}$.

225. After the ſame Manner, you may raiſe general Theorems when there are three unknown Quantities, x, y, and z,

and three Equations given; but more of this in another Plac
To conclude, we ſee by the above Rules, *That when there are a many ſimple Equations given as Quantities required, the Queſtion i limited, and the Quantities may be diſcovered and determined by thoſ Rules.*

226. But, *if there are more Quantities required, than Equation given, the Queſtion is not limited to determinate Quantities. And, if there are more Equations given, than Quantities required, it ma be impoſſible to find Quantities that may anſwer the Conditions of th Queſtion, becauſe ſome of them may be inconſiſtent with others.*

CHAP. IX.

A Collection of ſuch Queſtions as produce SIMPLE EQUATIONS.

227. THE Order of our Inſtitutions now brings us to con-ſider thoſe Queſtions which produce *Simple Equa-tions*; and theſe will be found not only proper Exerciſes for the Learner, but the firſt Example of the Uſe of this excellent Art; and here we ſhall follow the Method invented by Mr. *Ward*, on Account of its Perſpicuity and Eaſe, *viz.* that of numbering and regiſtering each Step of the Proceſs, as in the following *Solu-tions.*

QUESTION I.

228. If the Sum of two Numbers be 20, and their Diffe-rence 12; what are thoſe Numbers?

Firſt let	1	$x =$ the Greater Number	
And	2	$y =$ the Leſſer.	
Then	3	$x + y = 20$	per Queſtion.
And	4	$x - y = 12$	
Then (212)	5	$x = 20 - y$,	
And	6	$x = 12 + y$;	
Conſequently	7	$20 - y = 12 + y$.	
Then by	8	$20 - 12 = 2y = 8$; hence $y = 4$,	
And *per* Queſtion	9	$x + 4 = 20$,	
Therefore . . .	10	$x = 20 - 4 = 16$; See (Inſt. 221.)	

where you obſerve two general Theorems for any Queſtion of this Kind.

229.

QUESTION II.

229. Suppoſe there are two Numbers whoſe Sum is 32, and heir Ratio as 5 to 3. Query thoſe Numbers?

et the Numbers be	1	$x + y = 32$	The Data.
nd	2	$x : y :: 5 : 3$	
hen (138)	3	$3x = 5y$	
ivide by 3, . . .	4	$x = \frac{5}{3}y$	
lſo	5	$x = 32 - y$	
herefore . . .	6	$32 - y = \frac{5}{3}y$	
ultiply by 3, . .	7	$96 - 3y = 5y$	
nd	8	$96 = 8y$	
herefore	9	$y = \frac{96}{8} = 12.$	
ut by 5th . .	10	$x = 32 - y = 32 - 12 = 20.$	

he two Numbers therefore are 20 and 12, for $20 + 12 = 32$; nd $20 : 12 :: 5 : 3$, as required.

QUESTION III.

230. A Perſon being aſked how old he was, replied, that $\frac{3}{4}$ of s Age multiplied by $\frac{1}{12}$ of his Age, gives a Product equal to s Age. Query what was his Age?

ut	1	$x =$ to his Age.
hen *per* Queſtion	2	$\frac{3x}{4} \times \frac{x}{12} = x$
hat is	3	$\frac{3x^2}{48} = x$
hence (by 214)	4	$3x^2 = 48x$
hat is, (by 219)	5	$3x = 48$
onſequently (by 213)	6	$x = 16 =$ the Age required.

QUESTION IV.

231. A Perſon has ſix Sons, each of which is 4 Years older an the next younger Brother; and the Eldeſt is 3 Times as old the Youngeſt. What are their ſeveral Ages?

t their ſeveral Ages be . . .	1	x, $x + 4$, $x + 8$, $x + 12$, $x + 16$, and $x + 20$.
t *per* Queſtion	2	$3x = x + 20.$
hen, tranſpoſing,	3	$3x - x = 20.$
hat is . . .	4	$2x = 20.$
herefore . . .	5	$x = 10.$

their Ages are, 10, 14, 18, 22, 26, and 30, as required.

QUESTION V.

232. A Privateer running at the Rate of 10 Miles an Hour, diſcovers a Ship 18 Miles off making Way at the Rate of 8 Miles an Hour: It is demanded how many Miles the Ship can run before ſhe be overtaken?

Put . . . . 1 $x =$ Miles the Ship runs.
And . . . . 2 $y =$ Miles the Privateer runs.
Then by Suppoſition 3 $x : y :: 8 : 10$;
Therefore . . 4 $10x = 8y$, and $x = \frac{4}{5}y$.
But alſo . . . 5 $x = y - 18$;
Therefore . . 6 $\frac{4y}{5} = y - 18$,
Whence . . . 7 $4y = 5y - 90$
Conſequently . . . 8 $y = 90$; and $x = y - 18 = 72$.

To find the Time; you ſay, as 8 Miles : 1 Hour :: 72 Miles : 9 Hours.

QUESTION VI.

233. It is required to divide the Number 50 into two ſuch Parts, that $\frac{3}{4}$ of one Part being added to $\frac{5}{6}$ of the other, may make 40.

Put x y for the two Parts.

Then . . . . 1 $\frac{3x}{4} + \frac{5y}{6} = 40$.
Multiply by 4, 2 $3x + \frac{20y}{6} = 160$
Multiply by 6, 3 $18x + 20y = 960$
Therefore . . . 4 $20y = 960 - 18x$
Whence . . . . 5 $y = \frac{960 - 18x}{20} = 50 - x$
Hence . . . . . 6 $960 - 18x = 1000 - 20x$
Then . . . . . 7 $1000 - 960 = 2x = 40$
Conſequently . . . 8 $x = 20$; and therefore $y = 30$.

QUESTION VII.

234. Two Perſons, *A* and *B*, were talking of their Money: ſays *A* to *B*, lend me five Shillings of your Money, and I ſhall have juſt as much as you will have left: Says *B* to *A*, rather lend me five Shillings of your Money, and I ſhall then have juſt three Times as much as you will have left: How much Money had each?

Put $x = A$'s Money, and $y = B$'s.

Then, if *A* borrows,	1	$x + 5 = y - 5$
So that	2	$y = x + 10$
If *B* borrows, then	3	$y + 5 = 3 \times \overline{x - 5} = 3x - 15$
Therefore (*A*) . . .	4	$x + 10 + 5 = x + 15 = 3x - 15$
And ſo	5	$x + 30 = 3x$;
Conſequently	6	$2x = 30$, and $x = 15$s. *A*'s Money.
And	7	$x + 10 = y = 25$s. *B*'s Money.

QUESTION VIII.

235. A Perſon has three Debtors, *A*, *B* and *C*, whoſe particular Debts he has forgot; but thus much he could remember from his Accounts, that *A*'s and *B*'s Debts together amounted to 60 Pounds; *A*'s and *C*'s to 80 Pounds; and *B*'s and *C*'s to 92 Pounds: I demand the Particulars?

The Debts let be repreſented by x, y, and z.

Then	1	$x + y = 60$ } *per* Queſtion.
	2	$x + z = 80$ } *per* Queſtion.
	3	$y + z = 92$ } *per* Queſtion.
Then the 3—2d Step	4	$y - z = 92 - 80 = 12$
But 1ſt Step . . .	5	$y + z = 60$
Therefore	6	$2y = 60 + 12 = 72$; and $y = 36$.
Hence	7	$x + y = x + 36 = 60$; and $x = 60 - 36 = 24$.
And laſtly	8	$z = 92 - y = 92 - 36 = 56$.

So *A*'s Debt is 24*l.* *B*'s 36*l.* and *C*'s 56*l.*

QUESTION IX.

236. There is a certain Fiſh whoſe Head is 9 Inches; the Tail is as long as the Head and half the Back; and the Back is as long as both the Head and Tail together. I demand the Length of the Back, and of the Tail?

Put	1	$x =$ Length of the Back;
Then ſince . . .	2	$9 =$ Length of the Head,
We have	3	$x - 9 =$ the Length of the Tail.
But, by Suppoſition	4	$x - 9 = \frac{x}{2} + 9$;
Therefore	5	$2x - 18 = x + 18$;
And ſo	6	$2x - x - 18 = 18$
That is	7	$x - 18 = 18$, or $x = 36 =$ L. of the Back
And	8	$x - 9 = 27 =$ Length of the Tail.

So the whole Length of the Fiſh was $9 + 36 + 27 = 72$ Inches, and $27 = 9 + \frac{36}{2}$, and $36 = 9 + 27$.

QUESTION X.

237. One has a Leaſe for 99 Years; and being aſked how much of it was already expired, anſwered, that two Thirds of the Time paſt was equal to four Fifths of the Time to come; Quere the Times paſt, and to come?

Put . . . . 1 | $x =$ Time paſt.
Then . . . . 2 | $99 - x =$ Time to come.
But . . . . . 3 | $\frac{2x}{3} = \frac{4}{5}$ of $99 - x = \frac{396 - 4x}{5}$;
Therefore . . 4 | $2x = \frac{1188 - 12x}{5}$
Conſequently . . 5 | $10x = 1188 - 12x$
Hence . . . 6 | $10x + 12x = 22x = 1188.$
Therefore . . 7 | $x = \frac{1188}{22} = 54$ Years, the Time paſt.
And . . . . 8 | $99 - x = 45$, the Years to come.

Thus $\frac{2}{3}$ of 54 is 36, and $\frac{4}{5}$ of 45 is 36, as required.

QUESTION XI.

238. A Gentleman diſtributing Money among ſome poor People, found he wanted 10*s.* to be able to give 5*s.* to each, therefore he gives each 4*s.* only, and finds he has 5*s.* left. Query the Number of Shillings and poor People?

Let $x =$ Number of People, and $y =$ Number of Shillings.

Then by Suppoſition . . . . { 1 | $5x = y + 10$
 2 | $4x = y - 5$

Then . . . . 3 | $5x - 10 = y$
And . . . . 4 | $4x + 5 = y$,
Therefore . . 5 | $5x - 10 = 4x + 5$
Conſequently . . 6 | $5x - 4x = x = 15 =$ the poor People.
And therefore . . 7 | $y = 4x + 5 = 65 =$ the Shillings.

QUESTION XII.

239. Suppoſe the Diſtance between *London* and *Edinburgh* to be 360 Miles, and that a Courier ſets out from *Edinburgh* running at the Rate of 10 Miles an Hour: Another ſets out at the ſame Time from *London*, and runs the Rate of 8 Miles an Hour. It is required to know where they will meet?

Suppoſe the Courier from *Edinburgh* runs x Miles, and the other y Miles before they meet.

Then . . . 1 | $x + y = 360$ } by Suppoſition
2 | $x : y :: 5 : 4$ }
Therefore . . 3 | $4x = 5y$; and $x = \frac{5}{4}y$.
Alſo . . . . 4 | $x = 360 - y = \frac{5}{4}y$
Hence . . . 5 | $\frac{5}{4}y + y = 360$.
Therefore . . 6 | $9y = 1440$.
Conſequently 7 | $y = 160$.
8 | $x = 360 - y = 200$.

QUESTION XIII.

240. One ſets out from a certain Place and travels at the Rate of 7 Miles in 5 Hours, and 8 Hours after another ſets out from the ſame Place and travels the ſame Road at the Rate of 5 Miles in 3 Hours. How long and how far muſt the Firſt travel before he is overtaken by the Second.

Put . . . 1 | $x =$ Hours the 1ſt travelled.
Then . . 2 | $x - 8 =$ Hours the 2d travelled.
Now ſay, 3 | $5 : 7 :: x : \frac{7}{5}x =$ Whole Diſtance.
Alſo . . 4 | $3 : 5 :: x - 8 : \frac{5x - 40}{3} =$ Whole Diſtance.
Therefore 5 | $\frac{5x - 40}{3} = \frac{7}{5}x$
Whence 6 | $25x - 200 = 21x$.
Therefore 7 | $25x - 21x = 4x = 200$
Hence . . 8 | $x = 50$; and $x - 8 = 42$.
And ſo 9 | $\frac{7}{5}x = 70$ Miles they both travelled.

241. Theſe Examples may ſuffice for a mixt or numerical Analyſis; but Queſtions ſolved in a general Way by Symbols only, are much more uſeful ſince they then become general Canons or Theorems, and are applicable to every particular Caſe; whereas the other Way in Numbers is confined to one Caſe only; therefore we ſhall proceed to give a few Examples of general Solutions. Thus,

QUESTION XIV.

242. Two Couriers, *A* and *B*, ſet out from two Places diſtant from each other *d* Miles; of which the foremoſt *A* travels *p* Miles in *q* Hours, and *B* travels *r* Miles in *s* Hours; he is ſuppoſed to walk faſteſt, and the ſame Way with *A*. It is required

to know how many Miles each muſt travel before *B* can overtake *A*?

Put . . .	1	$x =$ Miles *A* travelled
	2	$y =$ Miles *B* travelled
Then alſo	3	$x = y - d$
And we have	4	$q : p :: 1 : \frac{p}{q} =$ Miles *per* Hours by *A*
Alſo	5	$s : r :: s\,\frac{r}{s} =$ Miles *B* walks *per* Hour
Then we have	6	$x : y :: \frac{p}{q} : \frac{r}{s}$
Hence	7	$\frac{xr}{s} = \frac{py}{q}$
Therefore	8	$qrx = psy$
Whence . . .	9	$\frac{psy}{qr} = x = y - d$
And ſo . . .	10	$psy = qry - qrd$
And	11	$psy + qrd = qry$
Whence . . .	12	$qrd = qry - psy$
Therefore	13	$\frac{qrd}{qr - ps} = y =$ Miles *B* travelled
And laſtly	14	$x = y - d =$ Miles travelled by *A*.

N. B. If $p = 8$, $q = 1$; $r = 10$, $s = 1$; and $d = 18$; then $y = 90$, $x = 72$, as in Queſtion 11. Thus *A* is the Ship, and *B* the Privateer that purſued her. But more generally yet;

QUESTION XV.

243. Let *A* be a moveable Body, which in Time *f* can deſcribe the Space *c*; and let *B* be another moving Body, which in the Time *g* can deſcribe the Space *d*; let the Diſtance from which they begin to move, be called *a* and *b*, the Difference of the Time.

In the Solution of this Problem, there will be two Caſes, the firſt is, when both the Bodies, *A* and *B*, tend towards the ſame Parts, or move the ſame Way. The ſecond Caſe is, when they tend towards contrary Parts, or meet each other. In both Caſes we ſuppoſe the Body *A* to be fartheſt from the Place where they come together.

CASE I.

244. Let that Diſtance of *A* be called x, from which take the Interval e, and there will remain $x - e$ for the Diſtance of *B*, from the ſaid Place where they come together. And ſince *A* paſſeth over the Space c in the Time of f, it will paſs over the Space x in the Time of $\frac{fx}{c}$, for it is $c : f :: x : \frac{fx}{c}$.

Alſo ſince *B* paſſeth over the Space d in the Time g, it will paſs over the Space $x - e$ in the Time $\frac{gx - ge}{d}$; ſince $d : g :: x - e : \frac{gx - ge}{d}$.

Now ſince the Difference of thoſe Times is ſuppoſed be h, that they may become equal; we muſt add h to the ſhorteſt Time, that is, (ſince *A* is ſuppoſed to move firſt) to the ſhorteſt Time of *B*, *viz.* to $\frac{gx - ge}{d}$; and then we have the Time equalled, *viz.* $\frac{gx - ge}{d} + h = \frac{fx}{c}$; and which reduced, gives $x = \frac{cge - cdh}{cg - df}$. But if the neareſt Body *B* begin to move firſt, h muſt be added to the ſhorter Time of *A*, viz. $\frac{fx}{c}$ and we have $\frac{fx}{c} + h = \frac{gx - ge}{d}$; which reduced gives $\frac{cge + cdh}{cg - fd} = x$.

CASE II.

245. If the Bodies, *A* and *B* meet; then if $x =$ Diſtance of the remoteſt Body *A* from the Place where they meet, as before, we have $e - x =$ Diſtance of *B* from the ſame Spot. And the Times in which they paſs over thoſe Spaces will be $\frac{fx}{c}$, and $\frac{ge - gx}{d}$, found as before. Then if *A* moves firſt, we have $\frac{ge - cx}{d} + h = \frac{fx}{c}$ and this Equation gives $\frac{cge + cdh}{cg + df} = x$.

But if B move firſt, we have $\frac{fx}{c} + h = \frac{ge - gx}{d}$; whence we get $\frac{cge - cgh}{cg + df} = x$.

EXAMPLE I.

246. The Sun deſcribes each Day a Degree in the Ecliptic, and the Moon Thirteen; and at a certain Time, when the Sun is in the Beginning of *Cancer*, ſuppoſe the Moon, three Days after, to be in the Beginning of *Aries*. It is required to know in what Part of the Ecliptic the next new Moon will happen?

In order to ſolve this Problem, we muſt conſider that A repreſents the Moon, and B repreſents the Sun, and that they both tend towards the ſame Part; and laſtly, that the Motion of A is ſhorter than that of B; therefore the Theorem $\frac{cge + cdh}{cg - df} = x$ muſt be uſed, in which the Values of the ſeveral Letters are known. Thus,

$c = 13$ Degrees, the Moon's Motion *per* Day.
$f = 1$ Day.
$d = 1$ Degree, the Sun's Motion *per* Day.
$g = 1$ Day.
$e = 90$ Degrees, the Diſtance of the Sun and Moon.
$h = 3$ Days, the difference in Time.

Hence the above Theorem is expreſſed in Numbers thus, $\frac{ege + cdh}{cg - df} = \frac{13 \times 1 \times 90 + 13 \times 1 \times 3}{13 \times 1 - 1 \times 1} = \frac{1209}{12} = 100\frac{3}{4}$, that is, $100\frac{3}{4}$ Degrees from the Beginning of *Aries*, ſo that the Conjuction happens in $10\frac{3}{4}$ Degrees of *Cancer*.

After the ſame Manner, A and B may repreſent the two Hands of a Clock, either ſetting out from the ſame Place at the ſame Time, or from different Places at different Times.

Again, by Theorem $\frac{cge + cdh}{cg - df} = x$, A and B may repreſent two Perſons ſetting out from two different Places, in order to meet, either at the ſame, or different Times, as in the former Queſtion, for *London* and *Edinburgh*, and the Diſtance travelled by each will be found by the Theorem, the ſame as was there determined. But theſe Things we ſhall leave as a proper Exerciſe for the young Learner.

CHAP. X.

Queſtions analytically ſolved, to illuſtrate the NATURE *and* REASON *of the ſeveral* RULES *of* ARITHMETIC, *in Vulgar Computation.*

247. WE ſhall here give the Solution of ſuch Queſtions, in an Algebraic Way, as will ſhew the *Rationale* of he Methods uſed in the ſeveral Operations of *Vulgar Arithmetic*, ſuch as the DOUBLE RULE OF THREE, RULES OF FELLOWSHIP, ALLIGATION, SIMPLE INTEREST, FALSE POSITION, &c. as our principal View in all theſe Things, is rather to make n intelligent than practical Artiſt, ſince many are ready enough t Practice, at the ſame Time they know little or nothing of the Reaſon of what they do.

I. DOUBLE RULE *of* THREE.

248. In all Queſtions of this Kind, there are three principal Terms which contain the Suppoſition, and three others in which the Queſtion lies; the three firſt are as follows:

A, the efficient Cauſe that produces an Effect;
T, the Time in which that Effect is produced;
E, the Effect produced in the given Time.

The other three Terms, which move the Queſtion, are of the ſame Kind, and therefore to be denoted by the ſame Letters *a*, , *e*, in ſmall Characters; hence, in order to raiſe Theorems or ſtating Queſtions in this Rule, we muſt reaſon as follows:

249. In equal Times, the Effects produced will always be proportioned to the efficient Cauſes: This is a ſelf-evident Axiom, or indiſputable Truth. Therefore, to expreſs this in Symbols, we muſt put $T = t$, and then we have this Analogy, $E : e :: A : a$; therefore $Ae = Ea$.

250. Another Axiom is, that when the efficient Cauſes are equal, the Effects which they produce will always be proportioned to the Times that are employed in producing them. That is n Symbols, if $A = a$, then $E : e :: T : t$, and therefore $Et = eT$.

251. But when neither the Times nor the efficient Cauſes ar equal, the Effects produced will be greater in Proportion to them both conjointly, that is, $E : e :: AT : at$, and therefore $atE = ATe$, from which general Theorem, if any five of thoſe Quantities are given, the Sixth may be found without any regard to the Proportions being *Direct* or *Inverſe*. And the three Terms of the Queſtion may be expreſſed by the three following Theorems, *viz.* Theo. I. $a = \frac{ATe}{Et}$. Theo. II. $e = \frac{atE}{AT}$. Theo. III. $t = \frac{eAT}{Ea}$.

EXAMPLE I.

252. Suppoſe 100*l.* in 12 Months, gain 4*l.* Intereſt, what Principal or Sum muſt be put out to gain 132*l.* in 66 Months? To anſwer this Queſtion, we have in Theorem I. the Value of the Symbols as follows:

$A = 100$, $T = 12$, $E = 4$, $t = 66$, $e = 132$, to find a, and the Theorem will ſtand in Numbers thus, $\frac{100 \times 12 \times 132}{4 \times 66} = a = 600l.$ the Anſwer.

EXAMPLE II.

If 2 Men in 3 Days will earn 4 Shillings, how much will 5 Men earn in 6 Days?

In this Caſe, $A = 2$, $E = 4$, $T = 3$, and $a = 5$, $t = 6$. To find e, for which Purpoſe we muſt make uſe of Theorem II. $\frac{atE}{AT} = e$, that is in Numbers $\frac{5 \times 6 \times 4}{2 \times 3} = 20$ Shillings, the Anſwer.

EXAMPLE III.

If for the Carriage of 300 Weight 40 Miles, I muſt pay 7 Shillings and 6 Pence, how far may I carry 500 Weight for 18 Shillings and 9 Pence?

In this Queſtion we have given $A = 300$, $E = 90$ Pence, and $T = 40$, $a = 500$, $e = 225$ Pence, to find t?

N. B. In Queſtions of this Sort, the Letter T, or *t*, may denote, not only *Time*, but alſo *Diſtance*, or any other Mode, or Way by which any Cauſe can produce its Effect, the Solution therefore of this Queſtion, is by Theorem III. *viz.* $\frac{eAT}{Ea} = t$,

which in Numbers, ſtands thus, $\frac{225 \times 30 \times 40}{90 \times 500} = 60$ Miles.

The RULE *of* FELLOWSHIP.

253. Suppoſe $\begin{cases} \text{A, B, C, D, \&c. Merchants trading in Company.} \\ a,\ b,\ c,\ d, \text{ \&c. Each Man's Money in Stock.} \\ w, x, y, z, \text{\&c. The Time it is employed by each.} \end{cases}$

Now ſince each Perſon's Share, of the Loſs or Gain, muſt be proportioned to the Money he advances, and alſo to the Time it is employed in Trade, therefore his Share will be as the Product of both multiplied together; that is to ſay, $aw =$ *A*'s Share, $bx =$ *B*'s Share, and ſo on.

Therefore let $\begin{cases} aw + bx + cy + dz, \text{ \&c.} = \text{S, the Sum of} \\ \quad \text{the Products of the Times and Stocks.} \\ \text{G} = \text{the whole Gain, Loſs, \&c. by Trading.} \end{cases}$

Then it is evident as S : G : : each of the above Products : Merchant's Share of Loſs or Gain reſpectively: That is, in Symbols as follows:

$$\left.\begin{array}{l} \text{Theorem I. } S : G :: aw : \frac{G}{S} \times aw = A\text{'s Gain.} \\ \text{Theorem II. } S : G :: bx : \frac{G}{S} \times bx = B\text{'s Gain.} \\ \text{Theorem III. } S : G :: cy : \frac{G}{S} \times cy = C\text{'s Gain.} \\ \text{Theorem IV. } S : G :: dz : \frac{G}{S} \times dz = D\text{'s Gain.} \end{array}\right\} \text{\&c.}$$

EXAMPLE.

254. Three Merchants *A*, *B*, and *C*, enter into Partnerſhip, thus;

A puts in 60*l.* for 8 Months, *B* puts in 75*l.* for 12 Months, and *C* puts in 80*l.* for 9 Months, with this joint Stock they Traffic and Gain a 150*l.* it is required to find each Man's Share proportioned to his Stock and Time.

$$\text{Firſt} \left\{\begin{array}{l} A\text{'s Stock } 60l. \times \ 8 = 480. \\ B\text{'s Stock } 75l. \times 12 = 900. \\ C\text{'s Stock } 80l. \times \ 9 = \underline{720.} \end{array}\right\} \text{Then ſay,}$$

The Sum of the Products = 2100.

As

$$\text{As } 2100:150 \begin{cases} ::480:34,\ 2857 = 34\ \overset{l.}{}\ 5\ 8\tfrac{1}{2}\ \text{A's Gain.} \\ ::900:64,\ 2857 = 64\ 5\ 8\tfrac{1}{2}\ \text{B's Gain.} \\ ::720:51,\ 42857 = 51\ 8\ 6\tfrac{3}{4}\ \text{C's Gain.} \end{cases}$$

N. B. When all the Stocks are employed an equal Time, the Theorems above will become more Simple, for then we neglect the Characters, w, x, y, z, as being each equal to Unity, in which Caſe thoſe Theorems give the Rules of what is uſually called SINGLE FELLOWSHIP, which, as it is ſo eaſy, we need not here farther inſiſt on.

The RULE *of* ALLIGATION.

255. Alligation is a Rule for compounding or mixing ſeveral Ingredients of different Sorts together, in any Manner or Proportion.

Let the Quantities to be compounded be a, b.
Their Prices ——— ——— ——— x, y.
The Compound which they make ——— c
And its Price ——— ——— ——— p
Then we have ——— ——— $a + b = c$
And alſo ——— ——— $xa + yb = pc$
Whence this Analogy, as $c : xa + yb :: 1 : p$.

That is in Words, *as the Sum of the Quantities is to the Sum of the Products of each by its Price, ſo is any one Part of the Compound or Mixture to the Price of that Part.*

EXAMPLE.

Suppoſe 20 Pounds of Tobacco at 9 Pence *per* Pound, were to be mixed with 12 Pound of Ditto at 14 Pence *per* Pound, what will a Pound of that Mixture be worth? Here $a = 20$, $b = 12$; $x = 9$, $y = 14$; and $a + b = c = 32$; then $ax = 180$, $by = 168$; and $ax + yb = 348$. Therefore $32 : 348 :: 1 :$ 10*d.* $\frac{3}{4}$, nearly, the Anſwer. This is called *Alligation Medial.*

256. When the Prices or particular Rates of the ſeveral Ingredients, and the Mixture and its Price are given, to find the particular Quantities of the Ingredients, it is called *Alligation Alternate*, as being the Reverſe of the foregoing.

If two Quantities only are concerned, then will the Equation be (as before) $xa + yb = pc$;

Then . . . . $yb = pc - xa$

And . . . . $b = \frac{pc - xa}{y}$

Alſo it is . . . . $a + b = c$

Therefore . . . . $b = c - a = \frac{pc - xa}{y}$

Conſequently . . . . . $cy - ya = pc - xa$

Then . . . . $cy + xa - ya = pc$

And . . . . $xa - ya = pc - yc$

And therefore . . . . $a = \frac{p - y}{x - y} \times c$.

Example.

257. How much Wine at 16 Pence a Pint, and of another Sort at 10 Pence a Pint, will compoſe a Mixture that ſhall be worth 12 Pence a Pint? Here $x = 16$, $y = 10$; and $p = 12$, and $c = 1$; therefore the Rule or Theorem is $a = \frac{p - y}{x - y} = \frac{12 - 10}{16 - 10} = \frac{2}{6} = \frac{1}{3}$; therefore $\frac{1}{3}$ of a Pint of the firſt, and $\frac{2}{3}$ of a Pint of the ſecond Sort muſt be taken.

N. B. If $y = 0$, then b will repreſent Water, whoſe Price is Nothing, and then $a = \frac{p}{x} = \frac{12}{16} = \frac{3}{4}$ of a Pint.

258. When more than two Quantities are to be compounded, and all unknown, the Queſtion will be *unlimited*, as there can be but two Equations, this will appear by the Steps of the following Proceſs.

Let	1	a, e, y, be three Ingredients.
And	2	r, s, t, their ſeveral Prices
Alſo let . . .	3	m, and p, be the Compound and its Price
Then	4	$a + e + y = m$
And	5	$ar + se + ty = pm$
Step 1ſt $-\ a$ . . .	6	$e + y = m - a$
$5 - ar$ . . .	7	$se + ty = pm - ra$
$6 \times$ by t . . .	8	$te + ty = tm - ta$
$7 - 8$ . . .	9	$se - te = pm - tm - ra + ta$

Therefore . . . 10 $e = \frac{p - t}{s - t} \times m - \frac{r - t}{s - t} \times a.$

Hence by 6th and 9th 11 $y + \frac{pm - tm - ra + ta}{s - t} = m - a$

And reducing, . . 12 $\left\{\begin{array}{l} sy - ty + pm - tm - ra + ta \\ = sm - sa - tm + ta. \end{array}\right.$

That is . . . 13 $sy - ty + pm - ra = sm - sa$

And . . . . 14 $sy - ty = sm - pm - sa + ra$

Conſequently . . . 15 $y = \frac{r - s}{s - t} \times a - \frac{p - s}{s - t} \times m.$

Now here it is evident from the 10th Step, that a muſt be leſs than $\frac{p - t}{r - t} \times m$; and from the laſt Step it appears, that a muſt be grearer than $\frac{p - s}{r - s}$. Therefore any Value of a may be taken between theſe two Limits.

EXAMPLE.

259. How much Tobacco at 2*s.* 8*d.* *per* Pound, and of another Sort at 20*d.* and of a third Sort at 16*d.* *per* Pound, muſt be taken to make a Mixture that ſhall contain 56 ℔, and may be ſold for 22*d.* *per* ℔, without Loſs or Gain?

Here $r = 32$, $s = 20$, $t = 16$; $m = 56$, and $p = 22$; alſo a is the Quantity worth 32*d.* *per* Pound, e that of 20*d.* and y that of 16*d.* *per* Pound, hence the greateſt Limit of a, will be $\frac{p - t}{r - t} \times m = \frac{22 - 16}{32 - 16} \times 56 = 21$, that is to ſay, a muſt be leſs than 21. Again, the leaſt Limit of a, will be $\frac{p - s}{r - s} \times m = \frac{22 - 20}{32 - 20} \times 56, = 9\frac{1}{3}$; therefore the leaſt Quantity of a, muſt be greater than $9\frac{1}{3}$. Hence 11 Anſwers may be obtained for this Queſtion, in whole Numbers, and an infinite Number in Fractions. Thus if the Values of a, be as in the firſt Column of the following Table, then the Values of e and y, by the 10th and 15th Steps, will be found ſuch as are contained in the 2d and 3d Columns.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

260. After the ſame Manner, if 4 Quantities or more are to be mixed together, you may proceed in the ſame Manner to find the Limits of the 1ſt and 2d, becauſe all the Quantities except the two laſt will admit of different Values, the Proceſs will be large and troubleſome; but if the Learner will purſue it, the Method is the ſame as before, and the Reaſon and Truth of the following Theorems will then appear, which I inveſtigated many Years ſince for perfecting this Part of Arithmetic.

$$\text{Let}\left\{\begin{array}{l}\mathrm{a} =\\ \mathrm{b} =\\ \mathrm{c} =\\ \mathrm{d} =\\ \mathrm{x} =\end{array}\right\}\text{Diff. between}\left\{\begin{array}{l}\text{the firſt Rate and the laſt}\\ \text{the 2d and laſt}\\ \text{the 3d and laſt}\\ \text{the 4th and laſt}\\ \text{the Mean Rate and the laſt.}\end{array}\right.$$

$$\text{Alſo let}\left\{\begin{array}{l}a =\\ b =\\ c =\\ d =\\ z =\end{array}\right\}\text{Diff. bet.}\left\{\begin{array}{l}\text{the Firſt}\\ \text{the 2d}\\ \text{the 3d}\\ \text{the 4th}\\ \text{the Mean}\end{array}\right\}\text{and the laſt Rate but one.}$$

$$\text{And}\left\{\begin{array}{l}p = \\ q = \\ n = \\ \mathrm{S} = \text{To the Sum of all the Ingredients, A + B + C}\end{array}\right.\left.\begin{array}{l}\text{Diff. between}\left\{\begin{array}{l}\text{Mean Rate and the ſecond}\\ \text{the firſt Rate and ſecond}\\ \text{the laſt but one and the laſt}\end{array}\right.\\ \end{array}\right.$$

+ D + E + F, *&c.*

261. Then if there be three Quantities to be compounded A + B + C = S, the Theorems are as follow. Theo. I. $\frac{\mathrm{Sx}}{\mathrm{a}}$ = A, greateſt. Theor. II. $\frac{\mathrm{S}p}{q}$ = A, leaſt. Theor. III. $\frac{\mathrm{Sx} - \mathrm{Aa}}{n}$ = B. Theor. IV. S − A − B = C.

262. But if four Quantities are to be mixed, *viz.* A, B, C, D, the firſt is found as before; the Theorems for B and C, are as follows.

Theor. V. $\frac{Sx - Aa}{b}$ = B, greateſt. Theor. VI. $\frac{Sz - A}{b}$ = B, leaſt. Theor. VII. $\frac{Sx - Aa - Bb}{n}$ = C; and then D is known of Courſe.

263. If there be five Ingredients, A, B, C, D, E, you hav the greateſt and leaſt Values of A and B as before; and for C and D as below.

Theor. VIII. $\frac{Sx - Aa - Bb}{c}$ = C, Greateſt.

Theor. IX. $\frac{Sz - Aa - Bb}{c}$ = C, Leaſt.

Theor. X. $\frac{Sx - Aa - Bb - Cc}{n}$ = D.

264. When there are ſix Quantities, A, B, C, D, E, F, the greateſt and leaſt Values of the three firſt are as in the laſt; and for D and E as follows.

Theor. XI. $\frac{Sx - Aa - Bb - Cc}{d}$ = D, Greateſt.

Theor. XII. $\frac{Sz - Aa - Bb - Cc}{d}$ = D, Leaſt.

Theor. XIII. $\frac{Sx - Aa - Bb - Cc - Dd}{n}$ = E.

And thus you proceed for any greater Number at Pleaſure.

265. As to the *Rule* of FALSE POSITION, as it is called, it can only be of Uſe to thoſe who are unacquainted with *Algebra*; and a Perſon may as ſoon learn the *Analytic Art*, as the *Reaſon* and *Operations* of this Rule without it; thus for Inſtance, in the *Single Rule* where *one Poſition* (or Suppoſition) only is required, as in the following Queſtion, *viz.*

266. Three Men, *A*, *B*, *C*, buy a Ship for 310*l.* 16*s.* of which *A* paid an unknown Sum; *B* paid $2\frac{1}{2}$ as much, and C $3\frac{1}{3}$ as much: How much did each Man pay?

Suppoſe *A* paid 48*l.* then *B* paid 48 × 2,5 = 120*l.* and *C* muſt pay 48 × 3,3 = 160*l.* But 48 + 120 + 160 = 328 inſtead of 310,75*l.*

Say therefore, As 328 : 48 : : 310,75 : 45,4756, &c.

Then *A* paid . . . . £. 45,4756
B paid $(45{,}4756 \times 2{,}5 =)$ . . . 113,689
C paid $(45{,}4756 \times 3{,}\mathfrak{z} =)$ 151,5853

roof is the Sum . . . . £.310,75

267. Now the Anſwer to this Queſtion is in a moſt direct and aſy Manner pointed out by Algebra; for let $x =$ Sum which *A* aid, then *B* paid $2{,}5\,x$, and *C* paid $3{,}\mathfrak{z}\,x$, but $x + 2{,}5\,x +$ $,\mathfrak{z}\,x = 310{,}75\,l.$ Therefore $\overline{1 + 2{,}5 + 3{,}\mathfrak{z}} \times x = 6{,}8\mathfrak{z}\,x$; ence $x = \dfrac{310{,}75}{6{,}8\mathfrak{z}} = 45{,}4756\,l.$ the Money *A* paid; then *B*'s nd *C*'s Part is found, as before. And as to the *Double Rule of alſe Poſition*, it is ſo troubleſome in Numbers, and ſo eaſy by lgebra, that we think it unworthy of an ingenious Artiſt, and hall leave it to the Indolent and Ignorant only, who generally ke the moſt Pains to the leaſt Purpoſe.

SIMPLE INTEREST.

268. Simple Intereſt is the Money paid for the Uſe or Loan f any Principal or Sum lent out for a given Time, at a certain ate *per Cent. per Annum.* Our Laws regulate this Matter; nd our Buſineſs is now to ſhew how the *Rules of Computation* e derived from *Theorems* inveſtigated by *Algebraic Reaſoning.* herefore,

et
P = any *Principal* or *Sum* put to Intereſt,
R = Ratio, or Rate *per Cent. per Ann.*
t = Time of Continuance at Intereſt,
A = Amount of the Principal and its Intereſt.

N.B. The *Ratio of the Rate* is the Simple Intereſt of 1 *l.* for e Year, at any given Rate. Thus, as $100 : 5 :: 1 : 0{,}05 =$ atio at 5 *per Cent. per Annum*; and $100 : 3\frac{1}{2} :: 1 : 0{,}035 =$ atio at $3\frac{1}{2}$ *per Cent. per Ann.*

269. Now R = the Intereſt of 1 *l.* for one Year,
2 R = the Intereſt of 1 *l.* for two Years,
3 R = the Intereſt of 1 *l.* for three Years,
And t R = the Intereſt of 1 *l.* for t Years.

But as 1 *l.* is to its Intereſt tR, for any given Time t, in the ſame Proportion as any Sum is to its Intereſt for the ſame Time. Hence the following Analogy:

$$1\,l. : tR :: P : tPR.$$

270. But the *Principal* and *Intereſt* added together is equal to the *Amount*, that is,

Theorem I. $tRP + P = A =$ the Amount.

Then Theorem II. $\dfrac{A}{tR + 1} = P =$ the Principal,

And Theorem III. $\dfrac{A - P}{tP} = R =$ the Ratio,

Laſtly, Theorem IV. $\dfrac{A - P}{RP} = t =$ the Time.

271. By theſe Theorems all Queſtious about Simple Intereſt are anſwered with the utmoſt Eaſe.

QUESTION. I.

What will 256 *l.* 10 *s.* amount to in three Years, one Quarter, two Months and eighteen Days, at 4 *per Cent. per Annum?*

In order to anſwer Queſtions in this Rule, the Parts of Integers, muſt be reduced to *Decimals*, eſpecially thoſe of Time, which may be readily done by obſerving,

$$\text{That one}\left\{\begin{array}{l}\text{Day is } \tfrac{1}{365} \text{ Part of a Year} = 0{,}00274\\ \text{Month is } \tfrac{1}{12} \text{ of a Year } = 0{,}08333\\ \text{Quarter is } \tfrac{1}{4} \text{ of a Year } = 0{,}25\end{array}\right.$$

$$\text{Then the Time is}\left\{\begin{array}{l}3 \text{ Years } = 3.\\ 1 \text{ Quarter} = 0{,}25\\ 2 \text{ Months} = 0{,}16667\\ 18 \text{ Days } = 0{,}04932\end{array}\right.$$

That is, $t = 3{,}46599$

Then $tR = 0{,}13864$; $tRP = 35{,}561057$; and laſtly, $tRP + P = 35{,}561057 + 256{,}5 = 292{,}061$, &c. $= 292$ *l.* 1 *s.* 2 *d.* ½ $= A$, the Amount required, as *per* Theor. I.

QUESTION II.

272. What Principal or Sum put to Intereſt at 4 *per Cent.* &c. will amount to 292 *l.* 1 *s.* 2 *d* ½ in 3 Years 1 Quarter, 2 Months, 18 Days?

Anſwer, per Theorem II. $\frac{A}{tR+1} = \frac{292,061}{0,13864+1} = \frac{292,061}{1,13864} = 256,5 = P = 256$ *l.* 10 *s.*

QUESTION III.

273. At what Rate *per Cent.* Intereſt will 256 *l.* 10 *s.* amount to 292 *l.* 1 *s.* 2 *d.* $\frac{1}{2}$ in 3 Years, 1 Quarter, 2 Months, 18 Days?

Anſwer, *per* Theorem III. $\frac{A-P}{tP} = \frac{292,061-256,5}{889,0234} = \frac{35,561}{889,0234} = 0,04 = R.$ So the Intereſt is 4 *per Cent.*

QUESTION IV.

274. In what Time will 256 *l.* 10 *s.* raiſe a Stock or amount to 292 *l.* 1 *s.* 2 *d.* $\frac{1}{2}$, at the Rate of 4 *per Cent.* Intereſt?

Anſwer, *per* Theorem IV. $\frac{A-P}{RP} = \frac{35,561}{10,26} = 3,46599 = t =$ 3 Years, 1 Quarter, 2 Months, 18 Days.

275. If it be conſidered, that t ſtands for any Time indifferently, as well a Part of a Year as a whole Year, or any Number of Years, it muſt appear very ſtrange for any one to imagine that, if 100 *l. in* 12 *Months gain* 4 *l. Intereſt, it will not gain* 2 *l. in* 6 *Months.* Yet this Mr. *Ward* affirmed (in his *Scholium*, p. 248) contrary to the Reaſon of Things, and to his firſt or *fundamental Theorem, viz.* $tRP + P = A$; for let $P = 100$ *l.* $R = 0,04$; $t = 0,5$; then $tRP + P = \overline{0,5 \times 0,04 \times 100 + 100} = 2 + 100 = 102$ *l.* $= A$, the true Amount in ſix Months; and therefore 2 *l.* muſt be the Intereſt for that Time.

N. B. What farther relates to Simple and Compound Intereſt muſt be deferred 'till after we have delivered the Doctrine of *Geometrical Progreſſion*, and *Quadratic Equations*, which we next proceed to.

CHAP.

CHAP. IX.

Of INVOLUTION.

276. BY *Involution* we mean a continual Multiplication of the ſame Quantity into itſelf; and the Products ariſing from thence, are called the *Powers* of that Quantity; and the Quantity itſelf is called the *Root*. Thus $a \times a = aa$; and a $\times a = aaa$, &c. So $ab \times ab = aabb$; and $aabb \times ab = aaabbb$; and ſo on. But when the ſame Quantity is often repeated, it is uſual to write it but once with a Figure above to ſhew how often it is repeated in any Product or Power; as thus for aa we write a^2; for aaa, a^3; for $aaabbb$, $a^3 b^3$; and the Figures are called the *Indices*, or Exponents of thoſe Powers, as they ſhew how many Times the Root has been involved from Unity as in the Table below.

277. 1 *Unity*.
$a =$ *Root*, or 1ſt Power.
$aa = a^2$, the *Square*, or 2d Power.
$aaa = a^3$, the *Cube*, or 3d Power.
$aaaa = a^4$, the *Biquadrate*, or 4th Power.
$aaaaa = a^5$, the *Surſolid*, or 5th Power.

And ſo on, as far as you involve the Root. Note, the Index of a^1 the firſt Power, being Unity, is never wrote; and we make $a^0 = 1$, or the firſt Term of the Series, when we would expreſs the Series compleat in Terms of a; as a^0, a^1, a^2, a^3, &c.

278. Hence it is evident, that theſe Powers are a Series of Terms in *Geometrical Proportion*, and their *Indices* the *Logarithms* of the Terms; the very ſame as we obſerved before of a like Series of Geometrical Terms in *Numbers* (130, 131, 132, &c.) Whence alſo it follows, that theſe Powers are *multiplied* and *divided* by *adding* and *ſubtracting their Indices* (132, 133.) Thus $a^2 \times a^3 = a^{2+3} = a^5$; and $a^2 b^2 \times a^3 b^3 = a^5 b^5$; and ſo in Diviſion $\frac{a^5}{a^2} = a^{5-2} = a^3$, and $\frac{a^5 b^5}{a^3 b^3} = a^{5-3} b^{5-3} = a^2 b^2$.

279. Hence likewiſe it appears, if a leſſer Power be divided by a greater the Index or Exponent of the Quotient muſt be negative:

ative: Thus $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, but $\frac{a^3}{a^5} = \frac{1}{a^2}$; there-ore alſo $\frac{1}{a^2} = a^{-2}$, (ſee 198).

Again, it is obvious, that $\frac{a}{a} = a^{1-1} = a^0$; but, $= \frac{1}{1} = 1$; therefore $a^0 = 1$; as before obſerved. In the ame Manner $\frac{1}{a} = \frac{a^0}{a} = a^{0-1} = a^{-1}$; and therefore 'tis vident $aaa.\ aa.\ a.\ 1.\ \frac{1}{a}.\ \frac{1}{aa}.\ \frac{1}{aaa}$, &c. may be thus ex-reſſed, $a^3.\ a^2.\ a^1.\ a^0.\ a^{-1}.\ a^{-2}.\ a^{-3}$, &c.

280. The *negative* Powers of *a*, are *poſitive* Powers of $\frac{1}{a}$ r a^{-1}; and are *multiplied and divided* by *adding and ſubtracting* heir Indices as before in the poſitive ones. Thus, $a^{-2} \times a^{-3}$ $= \left(\frac{1}{aa} \times \frac{1}{aaa} = \frac{1}{a^5} = \right) a^{-2-3} = a^{-5}$. And $\frac{a^{-5}}{a^{-3}} =$ $\div a^3 = a^{-5+3} = a^{-2}$; becauſe $\frac{1}{a^3}\Big)\,\frac{1}{a^5}\Big(= \frac{a^3}{a^5} =$ $= a^{3-5} = a^{-2}$ (279.)

281. Hence if a *poſitive Power* be multiplied by an equal *nega-ive Power* of any Quantity *a*, the Product will be *Unity*; *thus* $^{-3} \times a^3 = a^{3-3} = a^0 = 1$; for poſitive and negative owers deſtroy each other, and produce only Unity, which is o Power of *a* at all.

282. Suppoſe $\frac{a^{-3}}{a^{-5}} = \frac{a^{-3}}{a^{-3} \times a^{-2}} = \frac{1}{a^{-2}}$. But alſo is $\frac{1}{a^5}\Big)\,\frac{1}{a^3}\Big(= \frac{a^5}{a^3} = a^{5-3} = a^2$, therefore $\frac{1}{a^{-2}} =$ So that, in general, *any Quantity in the Denominator of a raction may be placed in the Numerator, with the Sign of the Index hanged.* Thus, $\frac{x}{a^3 b} = \frac{x}{b} \times \frac{1}{a^3} = \frac{x}{b} \times a^{-3} = \frac{x a^{-3}}{b}$; nd $\frac{x}{a^{-3} b} = \frac{x a^3}{b}$.

283. If (m) repreſent the indeterminate Index of any Pow of a that is poſitive, as a^m; then $\frac{1}{a^m} = a^{-m}$, will expreſs equal negative Power of a. And $a^m \times a^{-m} = a^{m-m} =$ $= 1$. Alſo, if a^n expreſs any other Power of a, then $a^m \times$ $= a^{m+n}$ is the Product of the two Powers a^m and a^n; and $= a^{m-n}$, is their Quotient.

184. To raiſe any ſimple Quantity to its 2d, 3d, 4th, Power, is, multiply its Exponent by 2, 3, 4, &c. (as *per* 134 thus the Square of a is $a^{1\times2} = a^2$; the third Power or Cube $a^{1\times3} = a^3$; the mth Power of a is $a^{1\times m} = a^m$; alſo the Square of a^3 is $a^{3\times2} = a^6$; the Cube of a^2 is $a^{2\times3} = a^6$; and the mth Power of a^2 is $a^{2\times m} = a^{2m}$, and the nth Power of a^m is $a^{m\times n} = a^{mn}$. The Square of ab is a^2b^2; the Cube is a^3b^3; the mth Power $a^m b^m$.

185. If there are Signs and Coefficients, the Signs muſt be ordered as directed in Multiplication, and the Coefficients involved, *viz.* ſquared, cubed, &c. as in the Examples below.

$$\begin{array}{rrl}
+2a & -3a & \text{Firſt Power.} \\
+2a & -3a & \\
\cline{1-2}
+4a^2 & +9a^2 & \text{Second Power.} \\
+2a & -3a & \\
\cline{1-2}
+8a^3 & -27a^3 & \text{Third Power.} \\
2a & -3a & \\
\cline{1-2}
16a^4 & +81a^4 & \text{Fourth Power.}
\end{array}$$

Where you obſerve the Powers of $+a$ are all affirmative; and thoſe of $-a$ are alternately negative and affirmative.

186. The Involution of a Compound Quantity is ſomewhat more operoſe, but performed after the ſame Manner, as you obſerve in the Binomial $a + b$ below.

$a +$

$$\begin{array}{l}
a + b = \text{Root, or firſt Power.} \\
a + b \\
\hline
a^2 + ab \\
\quad + ab + bb \\
\hline
a^2 + 2ab + b^2 = \text{Square, or 2d Power.} \\
a + b \\
\hline
a^3 + 2a^2b + ab^2 \\
\quad + a^2b + 2ab^2 + b^3 \\
\hline
a^3 + 3a^3b + 3ab^2 + b^3 = \text{Cube, or 3d Power.} \\
a + b \\
\hline
a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
\quad + a^3b + 3a^2b^2 + 3ab^3 3 + b^4 \\
\hline
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = \text{4th Power, \&c.}
\end{array}$$

287. The Powers of $a - b$ are raiſed in the ſame Manner, Regard being had to the Signs as follows.

$$\begin{array}{l}
a - b = \text{the Root, or firſt Power.} \\
a - b \\
\hline
a^2 - ab \\
\quad - ab + b^2 \\
\hline
a^2 - 2ab + b^2 = \text{Square, or ſecond Power.} \\
a - b \\
\hline
a^3 - 2a^2b + ab^2 \\
\quad - a^2b + 2ab^2 - b^3 \\
\hline
a^3 - 3a^2b + 3ab^2 - b^3 = \text{Cube, or 3d Power, \&c.}
\end{array}$$

Here it is eaſy to ſee, without proceeding farther, that the ſeveral Powers of this Binomial $a - b$ conſiſt of the ſame Terms as thoſe of the foregoing one $a + b$; the Difference being only in the Signs, where you obſerve the Sign is *Negative* wherever the Index of b is an *odd Number*, as b, b^3, *&c.* and affirmative when the ſaid Index is an *even Number* as b^2, b^4, *&c.*

288. In theſe Powers of $a+b$ or $a-b$ we are to take Notice of the following Particulars, *viz.* (1) In the firſt Term of any Power, the Quantity a has the Index of that Power; thus in the 2d Power, it is a^2; in the 4th Power it is a^4, &c. (2) That in the following Terms the Exponents of a decreaſe gradually by Unity; as in the 4th Power, they are $a^3\ a^2\ a^1$. (3) In the laſt Term, the Quantity a is not found at all. (4) That the ſame Things are true of the Quantity b, and Exponents, in a contrary Order; it is not found in the firſt Term; the Exponents increaſe by Unity from the 2d to the laſt Term; and that in the laſt Term the Exponent is the Index of the Power. (5) That therefore the Sum of the Exponents of both the Quantities in any Term is the ſame, and always equal to the Exponent or Index of the Power; thus in the 4th Power you find,

$$\text{The Exponents of } \begin{cases} a \quad 4 \quad 3 \quad 2 \quad 1 \quad 0 \\ b \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \end{cases}$$

Whence their Sums are $4+0=3+1=2+2=1+3=0+4=4$, the Exponent of the Power.

289. The Coefficients of the Terms alſo ariſe and proceed in a regular Order; and may be found for any Term, or all the Terms, of any Power of a Binomial by a *general Rule*; which, that it may be more obvious to the Reader, it may be proper to place all the Powers of $a+b$ as far as the 6th in one View as below:

$a+b$; firſt Power, or Root.
$a^2+2ab+b^2$; 2d Power.
$a^3+3a^2b+3ab^2+b^3$; 3d Power.
$a^4+4a^3b+6a^2b^2+4ab^3+b^4$; 4th Power.
$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$; 5th. Power.
$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$; 6th Power.

290. Here it may be obſerved, 1. That when the Number of Terms is odd, the Coefficient of the middle Term is the greateſt of all. 2. That the Coefficients on each Side the middle Term are the ſame, but in an inverſe Order. 3. That in thoſe Powers which conſiſt of an even Number of Terms, the Coefficients of the two middle Terms are the ſame; and alſo the Coefficients of the Terms on each Side are the ſame towards the Extremes. 4. That the Coefficients of the firſt and laſt Term being Unity, the

he Coefficient for any other Term may be found by the followng

RULE.

Multiply the Coefficient of the preceding Term by the Exponent of in that Term; and divide the Product by the Exponent of b *in the iven Term; the Quotient is the Coefficient required for that Term.*

Thus, for Example, the Coefficients of the Terms of the 6th Power are found *per* Rule $\frac{1 \times 6}{1} = 6 =$ the Coefficient of the 2d Term; then $\frac{6 \times 5}{2} = 15$, the Coefficient of the 3d Term: Again $\frac{15 \times 4}{3} = 20$ the Coefficient of the 4th or middle Terms. From all which it is plain, that the *Terms*, he *Coefficients*, and the *Signs* of the Terms of any Power of a Binomial may be known without the tedious Operation by a conſtant Multiplication.

291. But to repreſent this Matter more generally, let $a + b$ be raiſed to the Power m; then (by 288.) the Terms without their Coefficients will be a^m, $a^{m-1} b$, $a^{m-2} b^2$, $a^{m-3} b^3$, $a^{m-4} b^4$, $a^{m-5} b^5$, &c. continued 'till the Exponent of b becomes equal to m, for then the Series muſt end, as the Exponent of a will, in that Caſe, be nothing.

292. The Coefficients of the ſeveral Terms will be found by the foregoing Rule (190.) to be 1, m, $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, and ſo on till you have one Coefficient more than there are Units in m. So that we ſhall have $\overline{a + b}^m = a^m + m a^{m-1} b + m \times \frac{m-1}{2} a^{m-2} b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4$, &c.

293. By this general Theorem, or Series, you will find the Terms of any Power very readily by ſubſtituting the Exponent of the given Power every where for m. For Example, let the 4th Power of $a + b$ be required; then

$$\text{For the Coefficients}\left\{\begin{array}{c} m=4 \\ m\times\frac{m-1}{2}=6 \\ m\times\frac{m-1}{2}\times\frac{m-2}{3}=4 \\ m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}=1 \end{array}\right\}\begin{array}{l} m-1= \\ m-2= \\ m-3= \\ m-4= \end{array}$$

Therefore $\overline{a+b}^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

294. If any *Trinomial* or compound Quantity of three Terms, as $a+b+c$, be involved, the Square or 2d Power will be $a^2+2ab+bb+2ac+2bc+c^2$; where you will obſerve the Sum of the three firſt Terms is the Square of $a+b$; and the three laſt Terms are $2c\times\overline{a+b}+c^2$. Hence any Trinomial may be conſidered as compounded of a Binomial $a+b$ and a ſimple Quantity c; *viz.* $\overline{a+b+c}^2=\overline{a+b}^2+\overline{2c\times a+b+c^2}$. So that the Terms of the Square of any Trinomial may be known from thoſe of a Binomial without actual Involution. And in the ſame Manner it is ſhewn that the Cube of a Trinomial $\overline{a+b+c}^3=\overline{a+b}^3+\overline{3c\times a+b}^2+\overline{3c^2\times a+b}+c^3$.

295. We ſhall conclude with on Example of a Binomial in Numbers and Species together for the Sake of Illuſtration; let $a=10$, and $b=2$;

$$\begin{array}{lcccl}
\text{Then } a+b & = & 10+2 & = & 12 \\
a+b & = & 10+2 & = & 12 \\
\hline
a^2+ab & = & 100+20 & & 24 \\
+ab+b^2 & = & +20+4 & & 12 \\
\hline
a^2+2ab+b^2 & = & 100+40+4 & = & 144=\text{Square.} \\
a+b & = & 10+2 & & \\
\hline
a^3+2a^2b+ab^2 & = & 1000+400+40 & & 144 \\
a^2b+2ab^2+b^3 & = & 200+80+8 & & 12 \\
\hline
a^3+3a\ b+3ab^2+b^3 & = & 1000+600+120+8 & = & 1728=\text{Cube.}
\end{array}$$

CHAP.

CHAP. X.

Of EVOLUTION.

296. EVOLUTION is the Reverſe of Involution, and conſiſts in *reſolving of Powers into their Roots*, which Operation is called *Extraction* or *Evolution*. As *Involution* of Roots was performed by *Multiplication of Exponents*; (284.) ſo *Evolution* is made by *dividing the Exponent of the Power by the Index of the Root required.* (See the Reaſon of this in 135.) Thus the *Square Root* of a^2 is $a^{\frac{2}{2}} = a$; of a^3 it is $a^{\frac{3}{2}}$; of a^4, it is $a^{\frac{4}{2}} = a^2$; and the *Cube Root* of a^2 is $a^{\frac{2}{3}}$; of a^3 it is $a^{\frac{3}{3}} = a$; of a^4 it is $a^{\frac{4}{3}}$; of a^6, it is a^2; and ſo on. Alſo the Square Root of $a^2 b^2$ is ab; of $a^4 b^8 c^2$, it is $a^2 b^4 c$, and the *Cube Root* of $a^6 b^3$ is $a^2 b$; of $x^9 y^6 z^{12}$, it is $x^3 y^2 z^4$; and ſo of others.

297. From what has been ſaid with Reſpect to the *Signs* in Involution (279, 280.) it appears, *that any Power that has an affirmative Sign, may have an affirmative or negative Root, when the Index of that Root is an even Number.* Thus the Square Root of $+ a^2$ may be $+ a$, or $- a$; becauſe $+ a \times + a = a^2$, and alſo $- a \times - a = a^2$.

298. Alſo from thence likewiſe it follows *that no Root, whoſe Index is an even Number can be found for a Power with a negative Sign.* (See 285.) Thus the Square Root cannot be extracted from $- a^2$; and therefore ſuch Roots of *Negative Powers* are termed *impoſſible* or *imaginary*; and yet they are ſometimes to be conſidered, and will come into Uſe, as we ſhall ſee further on.

299. If the Root to be extracted is denoted by an *odd* Number; then the *Sign of the Root will be the ſame as that of the Power.* Thus the Cube Root of $- a^3$ is $- a$; and of $+ a^3$ it is $+ a$.

300. When the Index of the required Root will compleatly divide the Exponent of the Power propoſed; then that Root is only *a lower Power of the ſame Quantity.* Thus the Square Root of a^6 is $a^{\frac{6}{2}} = a^3$. and the Cube Root of $a^{12} = a^4$.

201. But otherwiſe, *the Root required will have a Fraction for its Exponent*; thus the Square Root of a^3 is $a^{\frac{3}{2}}$; and the Cube

Root of a^2 is $a^{\frac{2}{3}}$; alſo the Square Root of a itſelf is $a^{\frac{1}{2}}$; and ſo of others, as we ſhewed more generally before.

302. Any Quantity with a *fractional Exponent*, may be conſidered as *an imperfect Power*, or *Surd*; and the Symbol uſed to denote Extraction of *Surd Roots* is this $\sqrt{}$, within which the Power is placed, as $\sqrt{a^3}$, and without or over it, the Index of the Root, as $\sqrt[2]{a^3}$ denotes the Square Root of a^3; and $\sqrt[3]{a^5}$ is the Cube Root of a^5: So that $\sqrt[2]{a^3} = a^{\frac{3}{2}}$; and $\sqrt[3]{a^5} = a^{\frac{5}{3}}$; and univerſally, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

303. Theſe imperfect Powers or Surds, are *multiplied and divided*, like other Powers, *by their Exponents added and ſubtracted*: Thus $a^{\frac{1}{2}} \times a^{\frac{5}{2}} = a^{\frac{1}{2}+\frac{5}{2}} = a^{\frac{6}{2}} = a^3$; and $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}} = \sqrt[12]{a^{17}}$. And $a^{\frac{7}{2}} \div a^{\frac{3}{2}} = a^{\frac{7-3}{2}} = a^{\frac{4}{2}} = a^2$.

304. They are likewiſe *involved* and *evolved* in the ſame Manner as perfect Powers. Thus the Square of $a^{\frac{1}{2}}$ is $a^{\frac{1}{2} \times 2} = a^{\frac{2}{2}} = a$; the Square of $a^{\frac{3}{2}}$ is $a^{\frac{3}{2} \times 2} = a^{\frac{6}{2}} = a^3$. On the Contrary, the Square Root of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3} \div 2} = a^{\frac{2}{3 \times 2}} = a^{\frac{2}{6}} = a^{\frac{1}{3}}$; the Cube Root of $a^{\frac{3}{4}} = a^{\frac{3}{4 \times 3}} = a^{\frac{1}{4}}$.

305. As to the Roots of *Compound Powers* or Quantities, they are known from the very Form of thoſe Powers, if well conſidered, as they ſtand in the foregoing Tablet (289.) For if the Powers be of a *Binomial* Root $a + b$, and a *Square*, then it will conſiſt of three Terms, *viz.* the Square of a, the Square of b, and twice the Product of both, as a^2, $2ab$, bb; as to the Signs, if they are all affirmative, the Signs of each Part of the Root will be ſo; if the Product has a negative Sign, then the 2d Term of the Root will be *negative* or the Root will be $a - b$. If there are any Coefficients of the Squares a^2, and b^2, they will be Square Numbers, and their Roots will be the Coefficients reſpectively of the Terms of the Root a and b, and the Coefficient of the Product will be the Product of both. Therefore, unleſs a Quantity has all theſe Characters, it cannot be a Square. After the ſame Manner of Enquiry you may diſcover

If the propoſed Quantity be the *Cube* of any Binomial $a + b$, or $a - b$. It will be to little Purpoſe to ſay any Thing here of ſuch Powers as have Trinomial Roots, and indeed, unleſs any Quantity has apparently the above mentioned Criterions of a perfect Power, it will be in vain to attempt any particular Extraction of Roots at all. If any thing of this Kind is neceſſary to be done, it may be performed by the general *Formula* delivered for *involving* Roots (292.) and which will equally ſerve for evolving or extracting them, and that out of any Quantity, even *Surds*, and *infinite Series*.

306. For ſince in the Expreſſion $\overline{a + b}^{m}$, the Index ^{m}, if it be an *Integer*, ſhews the Power of $a + b$; ſo, on the Contrary, when it is a *Fraction*, it denotes the Root of $a + b$: for *a Root* is only a fractional Power of any Quantity. Therefore if $m = \frac{1}{2}$ then $\overline{a+b}^{m} = \overline{a + b}^{\frac{1}{2}} = \sqrt{a + b}$; Conſequently to extract the Square Root of $a + b$ is only to raiſe $a + b$ to the Power whoſe Index is $\frac{1}{2}$; then ſince $\overline{a + b}^{m} = a^{m} + m a^{m-1} b + m \times \frac{m-1}{2} a^{m-2} b^{2} + \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^{3}$, &c. and $m = \frac{1}{2}$ we ſhall have $a^{m} = a^{\frac{1}{2}}$

$$m a^{m-1} b = \tfrac{1}{2} a^{-\frac{1}{2}} b = \frac{b}{2 a^{\frac{1}{2}}}$$

$$m \times m \frac{-1}{2} a^{m-2} b^{2} = \tfrac{1}{2} \times -\tfrac{1}{4} a^{-\frac{3}{2}} b^{2} = -\frac{b^{2}}{8 a^{\frac{3}{2}}}$$

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^{3} = \tfrac{1}{2} \times -\tfrac{1}{4} \times \tfrac{1}{2} a^{-\frac{5}{2}} b^{3} = + \frac{b^{3}}{16 a^{\frac{5}{2}}}, \text{ \&c.}$$

So that $\overline{a + b}^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{b}{2 a^{\frac{1}{2}}} - \frac{b^{2}}{8 a^{\frac{3}{2}}} + \frac{b^{3}}{16 a^{\frac{5}{2}}}$, &c.

307. After the ſame Manner you will find, that $\overline{aa + xx}|^{\frac{1}{2}} = a + \frac{x^{2}}{2a} - \frac{x^{4}}{8a^{3}} + \frac{x^{6}}{16a^{5}}$, &c. And ſo if you would extract the Cube Root of $a^{3} + x^{3}$, you will find $\overline{a^{3} + x^{3}}|^{\frac{1}{3}} = a + \frac{x^{3}}{3a^{2}} - \frac{x^{6}}{9a^{5}} + \frac{5x^{9}}{81a^{8}} - \frac{10x^{12}}{243a^{11}}$, &c. And thus you may proceed for the propoſed Root of any other Binomial Quantity whatſoever.

CHAP.

CHAP. XI.

The RATIONALE *of Extracting the* SQUARE *a*
CUBE ROOT.

308. WHEN a Number is given, whoſe *Square Root* is to b extracted, the firſt Thing to be done is to *point* i that is, to place a Point over the firſt Figure and every other Fi gure afterwards; which Points reſolve the Number into Perio of two Figures each, unleſs the Number of Figures be odd, fo then the laſt Period can have but one Figure; the Reaſon o which is, that the Square of each of the nine Digits will produ but two Places of Figures; and 10 is the firſt Number who Square will produce *three Places* of Figures; and 100 is the fir that will produce *five Places* when ſquared; and ſo on. There fore the Points denote the Number of Figures in the Root. Th In $6\dot{4}$ there is but one Figure in the Root; in $\dot{1}4\dot{4}$ there are tw in $\dot{9}9\dot{8}5\dot{6}$ there are *three*; in $\dot{1}0\dot{0}2\dot{0}0\dot{1}$, there are four Figures i the Root; and ſo for any other Number. (123. 124.)

309. In extracting the *Cube Root*, the given Number is re ſolved into Periods of three Figures by the Points placed o the *firſt*, *fourth*, &c. becauſe the *Cube* of the greateſt Digit pro duces but *three* Places of Figures, or 10 is the firſt Numbe whoſe Cube makes *four* Places; and 100 cubed makes *ſeve* Places, and ſo on; therefore, over the *firſt*, *fourth*, *ſeventh*, *tenth*, &c. Figure we place a Point, to ſee how many Figure the Root will conſiſt of.

310. Now ſince there are always ſo many Places of Figur in the Root as there are Points, or Periods, in the given Numbe the Figures of each Place may be repreſented by Letters, *a*, *b*, *c*, &c. Thus, if there be but one Period, as 64, there will b but one Figure, *viz.* $8 = a$, and ſo $64 = a^2$, only. But in Number of two Periods as $\dot{1}4\dot{4}$, there will be two Figures i the Root, *viz.* $12 = 10 + 2 = a + b$. A Number of thr Periods, as 99856 will have a Root of three Places of Figu *viz.* $316 = 300 + 10 + 6 = a + b + c$, that is $a = 100$ $b = 10$, and $c = 6$. And thus you may proceed for any large Number of Places.

311. Again, ſince $\dot{1}4\dot{4} = \overline{a+b}^2 = a^2 + 2ab + b^2 =$
$^2 + \overline{2a+b} \times b = 100 + 44$; therefore $a^2 = 100$, and
$\overline{a+b} \times b = 44$; conſequently $a = 10$, and $2a = 20$;
nd ſince $\overline{20+b} \times b = 44$, it plainly ſhews that 20)44(2 =
, becauſe $\overline{20+2} \times 2 = 44$: Therefore the Work, in Sym-
ols and in Numbers, will ſtand as below.

$$\begin{array}{rll} & \dot{1}4\dot{4} & (\ 10 \\ & \underline{100} & \underline{+\ 2} \\ 20+2) & 44 & 12 = \text{Root.} \\ \times 2\ ) & \underline{44} & \\ & .\ . & \end{array} \qquad \begin{array}{rl} & a^2 + 2ab + bb\ (a \\ & \underline{a^2} \\ 2a+b\ ) & 2ab + bb \\ \times b\ ) & \underline{2ab + bb} \\ & .\ \ . \end{array}$$

312. When the propoſed Number has three Periods, as
$9\dot{8}5\dot{6}$, then there will be three Parts in the Root, $a + b + c$;
nd, therefore, $\overline{a+b+c}^2 = a^2 + 2ab + b^2 + 2ac +$
$bc + c^2 = \dot{9}9\dot{8}5\dot{6}$; here then $a^2 = 90000$, and $a = 300$, and
$a = 600$; conſequently $9856 = 2ab + b^2 + 2ac +$
$bc + c^2$; therefore 600) 9856 (10 = b; and now $\overline{2a+b}$
$b = 610 \times 10 = 6100$; therefore $9856 - 6100 = 3756$
$= 2ac + 2bc + c^2 = \overline{2a+2b+c} \times c = \overline{620+c} \times c$;
herefore 620) 3756 (6 = c; and ſo $626 \times 6 = 3756$. See
e Operation below.

$$\begin{array}{rlr} & \dot{9}9\dot{8}5\dot{6} & (\ 300 = a \\ & \underline{90000} & \underline{(\ \ 10 = b} \\ 600+10\ ) & 9856 & 310 = a+b \\ \times 10\ ) & \underline{6100} & \underline{6 = c} \\ 620+6\ ) & 3756 & 316 = a+b+c \\ \times 6\ ) & \underline{3756} & \\ & .\ .\ . & \end{array}$$

313. In the above Operation, the Cyphers being every where
nitted, the Work will be contracted, and appear in the com-

mon

mon Form, ſince then we take down but one Period at each Diviſion; for we muſt always make as many Diviſions as there are Places of Figures in the Root, and we can get no more than one at a Time.

$$
\begin{array}{rl}
 & \dot{9}9\dot{8}5\dot{6}\ (\ 316 \\
 & \underline{9} \\
61) & \cdot 98 \\
\times 1) & \underline{61} \\
626) & 3756 \\
\times 6) & \underline{3756} \\
 & \cdot\cdot\cdot\cdot
\end{array}
$$

By this analytic Proceſs, the Reaſon of extracting the Square Root of any Number, we preſume, is very evident, and particularly of all the Examples and Procedure in (123) and (124.)

314. And in the ſame Manner we demonſtrate the Rule for extracting the Cube Root. Thus, let the Cube Number $\dot{1}72\dot{8}$ be propoſed; there are two Points or Periods, the Root therefore will have two Places or Parts, *viz.* $a + b$, whoſe Cube is $a^3 + 3a^2b + 3ab^2 + b^3 = \dot{1}72\dot{8} = 1000 + 728$. Here 'tis evident $a^3 = 1000$, and ſo $a = 10$. Alſo $3a^2b + 3ab^2 + b^3 = 728$, and $3a^2 = 300$; therefore $300\ )\ 728\ (\ 2 = b$, whence we ſhall have $3a^2b = 600$, $3ab^2 = 120$, and $b^3 = 8$; conſequently $3a^2b + 3ab^2 + b^3 = 600 + 120 + 8 = 728$, the exact Remainder; therefore the Cube Root is $a + b = 10 + 2 = 12$.

315. Let another Example be the Number $\dot{1}382\dot{4} = a^3 + 3a^2 + 3ab^2 + b^3 = 13000 + 824$, and 'tis evident (from the Table of Powers, 122.) that the next Cube Number leſs than 13000 is 8000, the Root of which is $20 = a$; therefore $a^3 = 8000$, which ſubducted from the given Number leaves $5824 = 3a^2b + 3ab^2 + b^3$. Now $3a^2 = 1200\ )\ 5824\ (\ 4 = b$; therefore we have

$$3a^2b = 4800$$
$$3ab^2 = 960$$
$$b^3 = 64$$

$$5824 = 3a^3b + 3ab^2 + b^3.$$

And conſequently the Cube Root is $a + b = 20 + 4 = 24$. From all which it is evident the *common Rule*, given for extracting the Cube Root, is nothing more than the above *Analytical Proceſs in Words.*

Chap. XII.

The ANALYTIC *Doctrine of* ARITHMETICAL PROPORTION.

316. WE have already ſhewn that *Arithmetical Proportion* is that by which a Series of Numbers increaſe or decreaſe, by an *equal Quantity*, thro' the whole Progreſſion, (ſee Inſt. 96, 97.) and of which we have given ſufficient Examples in Numbers. We ſhall now proſecute that Doctrine farther, and in a more general Way by *Analyſis.* Thus, ſuppoſe the firſt Term of the Series be (a), and the *Ratio*, or Quantity, by which ach Term increaſes or decreaſes be (b), then will the ſecond Term be $a + b$, the third $a + 2b$, and the whole Series will ppear in this Form, *viz.*

If $\begin{cases} \text{Increaſing}, a.\ a+b.\ a+2b.\ a+3b.\ a+4b.\ a+5b, \&c. \\ \text{Decreaſing}\ a.\ a-b.\ a-2b.\ a-3b.\ a-4b.\ a-5b, \&c. \end{cases}$

317. Suppoſe the laſt Term of the Series be called x, then in he Series above we have the laſt Term $a + 5b = x$; then he firſt Term will be $a = x - 5b$; conſequently, if to this quation you conſtantly add b on both Sides, you will get the wo following Series, of the ſame Value in each Term, *viz.*

$$a. \quad a+b. \quad a+2b. \quad a+3b. \quad a+4b. \quad a+5b.$$
$$x-5b. \quad x-4b. \quad x-3b. \quad x\rightarrow 2b. \quad x-b. \quad x.$$

318.

318. If now you take the three laſt Terms of the ſecond Series, and place them in an inverted Order under the three firſt Terms of the firſt Series, and then add both together, you will have the Sum of the whole Series, thus

$$\begin{array}{l} a.\quad a+b.\quad a+2b \\ x.\quad x-b.\quad x-2b \\ \hline \end{array}$$

$$\text{Sum } a+x.\ a+x.\ a+x = 3a+3x = \overline{a+x} \times 3.$$

And ſince this will be the Caſe, let the Number of Terms be what it will, 'tis plain the *Sum total of any Series in Arithmetical Progreſſion is equal to the Sum of the firſt and laſt Terms, multiplied by half the Number of Terms.* Hence, if $n =$ the *Number* of Terms, and $s =$ the *Sum* of the whole Series, we have $s = \frac{n}{2} \times \overline{a+x}$, whence $a + x = \frac{2s}{n}$ and $x = \frac{2s}{n} - a$.

319. Again, ſince b is not in the firſt Term, the Coefficient of b will be always $n-1$; therefore the laſt Term will be $a + \overline{n-1} \times b = x = \frac{2s}{n} - a$, (by 999.) whence we get $s = \frac{2an + n^2 b - nb}{2} = \overline{a + \frac{nb-b}{2}} \times n$. Thus, for Inſtance, ſuppoſe the Series were $1 + 2 + 3 + 4 + 5 + 6$, &c. continued to 100; then $a = 1$, $b = 1$, and $n = 100$, whence the Sum of ſuch a Series will be $s = 1 + \frac{100-1}{2} \times 100 = 5050$.

320. Suppoſe the firſt Term of the Series be nothing, or a Cypher (0), then in the Equation above $x = \frac{2s}{n} - a$, ſince $a = 0$, we have $s = \frac{nx}{2}$ that is, *the Sum of ſuch a Series is equal to the greateſt Term multiplied by half the Number of Terms.* For Example, the Sum of the Series $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = \frac{10 \times 9}{2} = 45$. This is otherwiſe evident by adding the ſame Series to itſelf, with the Terms in an inverted Order, thus

$$\begin{array}{l} 0+1+2+3+4+5+6+7+8+9 \\ 9+8+7+6+5+4+3+2+1+0 \\ \hline \end{array}$$

Sum $9+9+9+9+9+9+9+9+9+9=10\times 9=90$.

The Half of which, therefore, is the Sum of the *ſingle Series*, *viz.* 45.

N. B. The Uſe of this Theorem will be found hereafter to be very conſiderable in many Parts of the mathematical and philoſophical Sciences.

CHAP. XIII.

Of GEOMETRICAL PROPORTION, or PROGRESSION.

321. WHEN a Series of Quantities encreaſe by a conſtant Multiplicator, or decreaſe by a common Diviſor, they are ſaid to be in *Geometrical Proportion*, as we have formerly obſerved in *Numbers* (ſee 98.) but ſhall here more generally treat of this Doctrine. The ſaid *Multiplier* or *Diviſor* is called the *Ratio* of the Series, which let be denoted by (r), then will the two Series be as follow

Increaſing; $a.\ ar.\ ar^2.\ ar^3.\ ar^4.\ ar^5.$ &c.

Decreaſing; $a.\ \frac{a}{r}.\ \frac{a}{r^2}.\ \frac{a}{r^3}.\ \frac{a}{r^4}.\ \frac{a}{r^5}.$ &c.

which latter Series is alſo thus expreſſed;

$$a\quad ar^{-1}\quad ar^{-2}\quad ar^{-3}\quad ar^{-4}\quad ar^{-5}\ \text{\&c.}$$

322. Now the Property of ſuch a Series is, *that the Product of the firſt and laſt Terms is equal to the Product of any two others equally diſtant from them; or to the Square of the middle Term, if the Number of Terms be odd.* Thus if we take ſix Terms, we have $a\times ar^5=ar\times ar^4=ar^2\times ar^3=a^2 r^5$. Or, if we take but five Terms, we ſhall have $a\times ar^4$

$= ar \times ar^3 = ar^2 \times ar^2 = a^2 r^4$, and ſo on. Whence all the other Properties of ſuch a Series, mentioned in Chap. IX. of Arithmetic, are eaſily deduced.

323. Suppoſe the laſt Term of the Series be called y, as $ar^5 = y$, then the firſt Term will be $a = \frac{y}{r^5} = yr^{-5}$ (in the increaſing Series;) the ſecond Term yr^{-4}, the third will be yr^{-3}, and ſo on; ſo that the Series will be doubly expreſſed as below, *viz.*

$$\begin{array}{cccccc} a & ar & ar^2 & ar^3 & ar^4 & ar^5, \text{\&c.} \\ yr^{-5} & yr^{-4} & yr^{-3} & yr^{-2} & yr^{-1} & y. \end{array}$$

324. Now 'tis evident the Product of the firſt Term in the firſt Series, and laſt Term in the other, will be equal to the Product of any other two Terms, taken at an equal Interval from theſe; that is, $a \times y = ar \times yr^{-1} = ar^2 \times yr^{-2} = ar^3 \times yr^{-3}$, &c. (by 138.)

325. In either Way of expreſſing ſuch a Series, we obſerve, *the Sum of all the Terms, except the firſt, is equal to the Sum of all, except the laſt, multiplied by the common Ratio* (r); for $ar + ar^2 + ar^3 + ar^4 + ar^5$, &c. $= \overline{a + ar + ar^2 + ar^3 + ar^4} \times r$; and therefore let $s =$ the Sum of the Series, and we have $s - a = \overline{s - y} \times r$, (which is the above Theorem in Symbols) and conſequently $s - a = sr - yr$. Whence $sr - s = yr - a$, and ſo $s = \frac{yr - a}{r - 1}$.

326. Since the Exponent of r begins from the ſecond Term, it will always be leſs by 1 than the Number of Terms n; and therefore in the laſt Term it will be $n - 1$; ſo that $y = ar^{n-1} = \frac{ar^n}{r} \because yr = ar^n$, and ſo $s = \left(\frac{yr - a}{r - 1} =\right) \frac{ar^n - a}{r - 1}$ Wherefore a, r, and n being given, the Sum of the Series s is eaſily found.

327. In any encreaſing Series, if the firſt Terms be a. ar. ar^2. ar^3, &c. and (y) the laſt Term, then will the four laſt Terms of the Series be yr^{-3} yr^{-2} yr^{-1} y; and thus both

Ends of the Series may be expreſſed, without the intermediate Terms. Thus

$$a.\ ar.\ ar^2.\ ar^3,\ \&c.\quad yr^{-3}.\ yr^{-2}.\ yr^{-1}.\ y.$$

And in a decreaſing Series they will ſtand thus,

$$a.\ ar^{-1}.\ ar^{-2}.\ ar^{-3},\ \&c.\quad yr^3.\ yr^2.\ yr.\ y.$$

In this Caſe we have $s - a = \overline{s - y} \div r$, or $rs - ra = sy$, nd ſo $s = \frac{ra - y}{r - 1} = \frac{yr^n - y}{r - 1}$ (ſee 325.)

328. If the Number of Terms in ſuch a Series be ſuppoſed nfinite, we ſhall have $y = o$; for ſince $n = 8$,* $rn - 1$ will be infinite alſo; and *a finite Quality divided by an infinite ne is nothing*, therefore $y = \frac{a}{r^{n-1}} = 0$; and conſequently the Sum of ſuch a decreaſing Series will be $s = \frac{ar}{r - 1}$ which is a inite Sum, though the Number of Terms be infinite. Thus $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \&c. = \frac{1 \times 2}{2 - 1} = 2$; and $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \&c. = \frac{1 \times 3}{3 - 1} = \frac{3}{2} = 1\frac{1}{2}$.

Theſe Theorems alſo we ſhall hereafter find of very great Uſe in Philoſophy.

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* We ſhall uſe the Symbol 8 for *Infinity*, when there is Occaſion.

CHAP. XIV.

Of the Nature, Geneſis, and Roots of QUADRATIC EQUATIONS.

329. A *Quadratic Equation* properly ſignifies no more than the *Square of a ſimple Equation*. Thus the Square of the ſimple Equation $x - b = 0$, $(\overline{x - b} \times \overline{x - b}) = x^2 - 2bx + b^2 = 0$, which is a *Quadratic Equation*; whence

all Equations where the higheſt Power of the unknown Qua
tity is a *Square*, or the ſecond Power, are call'd *Quadrati*
however they may differ from this in other Parts. For we a
here to obſerve, that this is the moſt ſimple Form of a Qu-
dratic, whoſe Root conſiſts of two Parts, or Quantities, $x -$
$b = 0$; this Root once obtained, gives the Value of x, vi
$x = b$.

330. All other Forms of Quadratics are Rectangles mad
of two different Roots, or Values of x, as $\overline{x - a} = 0 \times \overline{x -}$
$= 0$ makes $x^2 - \overline{a + b} \times x + ab = 0$; where it is obviou
the Coefficient of the ſecond Term is the Sum of the two Roots, vi
a + b; *and the third Term is the Rectangle of both, vix.* ab.

331. But now to determine the Signs by which the Te
will be in all Caſes connected, we muſt conſider the Quality o
the Roots, *viz.* whether they are *Affirmative* or *Negative.* I
the Equations above, the Roots are both affirmative; fo
ſince $x - a = 0$, and $x - b = 0$, it will be $x = a$, and $x =$
Let the Sum of the Roots be $a + b = s$, and the Rectangl
$ab = r$, then when both the Roots are affirmative, the For
of the Quadratic Equation will be $x^2 - sx + r = 0$.

332. If both the Roots are negative, *viz.* $x = -a$, and
$= -b$, then $x + a = 0$, and $x + b = 0$, and the Rectangl
of both Equations gives $x^2 + ax + bx + ab = 0$; that i
the Form in all ſuch Caſes is $x^2 + sx + r = 0$.

333. If one Root be affirmative, as $x = a$, and the othe
negative, $x = -b$, then the Equations $x - a = 0$, and $x +$
$b = 0$, multiplied together, produce $x^2 - ax + bx - ab =$
0; now as the Root a or b is the greater Quantity, the Sig
of the ſecond Term will be − or +, and being ambiguous
it is cuſtomary to expreſs both the Signs, thus $x^2 \mp sx -$
$= 0$; in this Form the laſt Term is always *negative*; and
= the Difference of the Roots.

334. Hence by obſerving the Signs of the Equation, yo
will readily diſcover when the Roots are affirmative or nega-
tive; for if the *Signs change alternately* from + to −, and from
− to +, as in the firſt Form, (331.) then both the Roo
are *affirmative*; but if the *Signs are all affirmative*, the Roo
are both negative, as in the ſecond Form, (332.) And

the

ere is but one Change of the Signs as in the 3d Form, (where ere are either +, +, —, or +, —, —,) there will be one firmative Root, and the other negative of Courſe.

335. There is another Diſtinction of the Roots of Equations, z. into *real* and *imaginary*, or *poſſible* or *impoſſible*. Thus in the quation $x^2 - a^2 = 0$, both the Roots are *real*; for ſince $= a^2$, it will be either $x = +a$, or $x = -a$, as appears (297.) But the Equation $x^2 + a^2 = 0$, has no poſſible oot at all; for here $x^2 = -a^2$, and $x = \sqrt{-a^2}$, which is npoſſible, as appears by (298.) The Roots, therefore, of a uadratic Equation are both *poſſible* or *real*, or both *impoſſible* or *aginary* together.

336. When a Queſtion is properly ſtated, in algebraic Terms, d produces a Quadratic Equation, it will always be of one the three Forms above-mentioned, *viz.*

$$\text{ither}\left\{\begin{array}{l} 1.\ x^2 - sx + r = 0 \\ 2.\ x^2 + sx + r = 0 \\ 3.\ x^2 \mp sx - r = 0 \end{array}\right\}\text{Which are all rectangular Quadratics. (330.)}$$

337. Now if in each of theſe we tranſpoſe the known Quan- y r, they will appear in the Form of an imperfect Square.

$$\text{hus}\left\{\begin{array}{l} 1.\ x^2 - sx = -r \\ 2.\ x^2 + sx = -r \\ 3.\ x^2 \mp sx = r \end{array}\right\}\text{The Form of a perfect Quadratic being } x^2 + 2bx + bb = 0.$$

338. In order then to reſolve a Quadratic Equation, the 1ſt hing to be done is to *complete the Square of it*, which is ea- y done, if we only obſerve, that the third Term wanting to ake it complete, *is the Square of half the Coefficient of the ſe- nd Term.* Thus in the complete Square $x^2 - 2bx + bb$, the ird Term bb is the Square of half the Coefficient $2b$ of the cond Term. Therefore ſince the Coefficient of the ſecond erm in all the foregoing Forms is s, the Half of which is $\frac{s}{2}$ d the Square of that Half is $\frac{ss}{4}$; which Square, therefore, makes e third Term to be added to each Side of the foregoing Equa- ns, to make each a complete Square on that Side where the

unknown Quantity x is; and when this Addition is made, thoſe Equations will ſtand as below, *viz.*

$$\left.\begin{array}{l} 1.\ x^2 - sx + \frac{1}{4}s^2 = \frac{s^2}{4} - r \\ 2.\ x^2 + sx + \frac{1}{4}s^2 = \frac{s^2}{4} - r \\ 3.\ x^2 \mp sx + \frac{1}{4}s^2 = \frac{s^2}{4} + r \end{array}\right\} \text{In the Form of perfect Squares, whoſe Root is } x \pm \tfrac{1}{2}s.$$

339. If now you extract the Root from theſe Equations, we ſhall have

$$\left.\begin{array}{l} 1.\ x - \frac{s}{2} = \sqrt{\frac{s^2}{4} - r} \\ 2.\ x + \frac{s}{2} = \sqrt{\frac{s^2}{4} - r} \\ 3.\ x \mp \frac{s}{2} = \sqrt{\frac{s^2}{4} + r} \end{array}\right\} \text{And by tranſpoſing } \frac{s}{2} \left\{\begin{array}{l} 1.\ x = \sqrt{\frac{s^2}{4} - r} + \frac{s}{2} \\ 2.\ x = \sqrt{\frac{s^2}{4} - r} - \frac{s}{2} \\ 3.\ x = \sqrt{\frac{s^2}{4} + r} \pm \frac{s}{2} \end{array}\right.$$

340. Since the Equation in any Form always contains two Roots, it is evident the Value of x muſt ever be expreſſed, in *two Parts or Terms*, ſince their Sum gives one Root or Value of x, and their Difference gives the other, which is the Reaſon why you ſee them always connected with the double Sign $\pm$.

341. But to put this Matter into a clearer Light, let us inveſtigate the two Roots, a and b, of a *Rectangular Equation*, ſeparately in Numbers, which we can do by having the Sum given, *viz.* $a + b = s$, and their Product $ab = r$. (228.) Thus ſuppoſe $x = a = 5$, and again $x = b = 3$, then $\overline{x - a} = 0 \times \overline{x - b} = 0$, gives $x^2 - ax - bx + ab = 0$, or $x^2 - 8x + 15 = 0$.

Here $a + b = 8$
And $ab = 15$

Therefore $b = 8 - a$
And $b = \frac{15}{a} = 8 - a$
And ſo $15 = 8a - a^2$
Tranſpoſing all the Terms $a^2 - 8a = -15$
And compleating the Square $a^2 - 8a + 16 = 16 - 15 = 1$
Extract the Root $a - 4 = \sqrt{1}$
And ſo $a = {}^{+}\sqrt{1} + 4 = 5$
And of Courſe $b\ (= 8 - a) = -\sqrt{1} + 4 = 3$

That is in Words, *The Sum of the Radical Quantity, and Half the Coefficient of the middle Term, is the greater Root* a; *and Half the Coefficient leſs the Radical Quantity, is the leſſer Root* b.

342. Before we leave this Subject, it will be proper to obſerve, there is yet a more ſimple Form by which the Root may be expreſſed, than thoſe in (339.) For ſince $\sqrt{\frac{s^2}{4} - r} = \sqrt{\frac{s^2 - 4r}{4}} = \sqrt{\frac{1}{4} \times s^2 - 4r} = \sqrt{\frac{1}{4}} \times \sqrt{s^2 - 4r} = \frac{1}{2} \times \sqrt{s^2 - 4r} = \frac{\sqrt{s^2 - 4r}}{2}$; therefore $\sqrt{\frac{s^2}{4} - r} + \frac{s}{2} = \frac{\sqrt{s^2 - 4r} + s}{2}$. Hence the Root of a Quadratic Equation will be either $\frac{\sqrt{s^2 - 4r} \pm s}{2}$ in the two firſt Forms, or $\frac{\sqrt{s^2 + 4r} \pm s}{2}$ in the third. Or more particularly, for every Form of a Quadratic Equation, the Root will be as in the following Table.

EQUATIONS.	ROOTS.
1. $x^2 - sx + r = 0$	$\frac{\sqrt{s^2 - 4r} + s}{2}$

EQUATIONS,	ROOTS.
2. $x^2 + sx + r = 0$	$\frac{\sqrt{s^2 - 4r} - s}{2}$
3. $x^2 - sx - r = 0$	$\frac{\sqrt{s^2 + 4r} + s}{2}$
4. $x^2 + sx - r = 0$	$\frac{\sqrt{s^2 + 4r} - s}{2}$

CHAP. XV.

A Collection of QUESTIONS *producing* QUADRATIC EQUATIONS.

343. IN this Place we mean the Solution of ſuch Queſtions or Problems, as involve not the pure *Square* of the unknown Quantity x, as xx; but the Sum or Difference of this Square, and the Rectangle of the Quantity x, and ſome known Quantity s, as $xx \pm sx$; and theſe are called *Adfected Quadratic Equations:* All of which are to be reduced to one or other of the *four Terms* in the laſt Article, and then the ſought Quantity x is known, and the Queſtion ſolved, as will be illuſtrated by the following Examples.

QUESTION I.

344. What two Numbers are thoſe whoſe Sum is 20, and their Product 36?

SOLUTION.

Let the two Numbers be x, y.

Then their Sum is $x + y = 20$.

Their Product $xy = 36$.

$$\text{Then we have } \begin{cases} y = 20 - x \\ y = \frac{36}{x} \end{cases}$$

$$\text{And conſequently } \frac{36}{x} = 20 - x$$

$$\text{Which gives } xx - 20x = 36$$

Therefore $x^2 - 20x - 36 = 0$, which is a Quadratic of the irſt Form, where $s = 20$, $r = 36$, and $x = \frac{\sqrt{s^2 - 4r} + s}{2}$ $= \frac{\sqrt{400 - 144} + 20}{2} = 18$; and ſo $y = 2$: Thus the Sum 8 + 2 = 0, and the Product $18 \times 2 = 36$.

QUESTION II.

345. What two Numbers are thoſe whoſe Difference is 24, nd the Quotient of the greater divided by the Square of the eaſt is 3?

SOLUTION.

Let the two Numbers be x and y

Then (per Queſtion) $x - y = 24$

And $\frac{x}{yy} = 3$

Which gives this Equation $x = 3y^2$, and $y = \sqrt{\frac{x}{3}}$

Therefore we have $x - 24 = y = \sqrt{\frac{x}{3}}$

Whence, by ſquaring, $x^2 - 48x + 576 = \frac{x}{3}$

Therefore, $3x^2 - 164x + 1728 = x$

And tranſpoſing, $3x^2 - 145x + 1728 = 0$

And dividing by 3, $x^2 - 48.3x + 576 = 0$

Here then, $s = 48.3$; $r = 576$; and (per Form firſt) $x = \frac{\sqrt{s^2 - 4r} + s}{2} = 27$, and therefore $y = 3$.

QUESTION III.

346. There are two Numbers, the Sum of their Squares is 2368, and the greater of them is in Proportion to the leſs as 6 to 1, What are the Numbers?

SOLUTION.

Let $a =$ the greater Number, $e =$ the leſſer, and $z = 2368$.

Then

Then	1	$aa + ee = z$	by the Queſtion
And	2	$a : e :: 6 : 1$	
$2 \therefore$	3	$1a = 6e$	
$3 \odot^{2}$	4	$aa = 36ee$	
$1 - 4$	5	$ee = z - 36ee$	
$5 + 36ee$	6	$37ee = z$	
$6 \div \overline{37}$	7	$ee = \frac{z}{37} = 64$	
$7\, \text{ш}^{2}$	8	$e = \sqrt{\frac{2}{37}} = 8$	
$8 \times \overline{6}$	9	$6e = 6\sqrt{\frac{z}{37}} = 48$	
3 and 9	10	$a = 48$	

Proof:

If $a = 48$
And $e = 8$
$aa = 2304$
$ee = 64$

$aa + ee = 2368$
and $48 : 8 :: 6 : 1$

QUESTION IV.

247. There are three Numbers in continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the two Extreams?

SOLUTION.

That is, ſuppoſe a, m, e, in continued Proportion, and $m = 72$.

Then	1	$a + e = 5156 = s$	by the Queſtion
	2	$a : m :: m : e$	Quære a, e, &c.
$2 \because$	3	$ae = mm$	
$1 \odot^{2}$	4	$aa + 2ae + ee = ss$	
3×4	5	$4ae = 4mm$	
$4 - 5$	6	$aa - 2ae + ee = ss - 4mm$	
$6\, \text{ш}^{2}$	7	$a - e = \sqrt{ss - 4mm}$	
$1 + 7$	8	$2a = s + \sqrt{ss - 4mm}$	
$8 \div 2$	9	$a = \frac{s + \sqrt{ss - 4mm}}{2} = 108$	Or $a = 48$
$1 - 9$	10	$e = \frac{s - \sqrt{ss - 4mm}}{2} = 48$	$e = 108$

QUES-

QUESTION V.

348. There are three Numbers in continued Proportion, their m is 74, and the Sum of their Squares is 1924: What are oſe Numbers?

SOLUTION.

That is, *a*, *e*, *y*, are in continued Proportion.

Then	1	$a+e+y=s=74$	Quære *a*, *e*, *y*.
	2	$aa+ee+yy=z=1924$	
	3	$a:e::e:y$	
5 ∵	4	$ay=ee$	
1 − *e*	5	$a+y=s-e$	
2 − *ee*	6	$aa+yy=z-ee$	
4 × 2	7	$2ay=2ee$	
6 + 7	8	$aa+2ay+yy=z+ee$	
$5 \circledcirc 2$	9	$aa+2ay+yy=ss-2se+ee$	
8 and 9	10	$z+ee=ss-2se+ee$	
10 ±	11	$2se=ss-z$	
1 ÷ 2*s*	12	$e=\dfrac{ss-z}{2s}=24$	
5	13	$a+y=s-e=50$	
$13 \circledcirc 2$	14	$aa+2ay+yy=2500$	
4 × 4	15	$4ay=4ee=2304$	
4 − 15	16	$aa-2ay+yy=196$	
16 ш^{2}	17	$a-y=\sqrt{196}=14$	
3 + 17	18	$2a=50+14=64$	
18 ÷ 2	19	$a=32$	Or $a=18$
3 − 19	20	$y=50-32=18$	$y=32$

QUESTION VI.

349. There are three Numbers in Arithmetical Progreſſion, he firſt being added to twice the 2d, and three Times the 3d, heir Sum will be 62; and the Sum of all their Squares is 275: What are thoſe Numbers?

SOLUTION.

uppoſe	1	*a*, *e*, *y*, in Arithmetical Progreſſion	
And	2	$a+2e+3y=62$	by the Queſtion
	3	$aa+ee+yy=275$	
Then	4	$a+y=2e$ (per Article 136.)	

$2 - 4$	5	$2e + 2y = 62 - 2e$
$5 \div 2$	6	$e + y = 31 - e$
$6 - e$	7	$y = 31 - 2e$
$4 - 7$	8	$a = 41 - 31$
$8 \circledcirc^{2}$	9	$aa = 16ee - 248e + 961$
$7 \circledcirc^{2}$	10	$yy = 961 - 124e + 4ee$
$9 + 10$	11	$aa + yy = 20ee - 372 + 1922$
$3 - 11$	12	$ee = 372e - 20ee - 1647$
$12 + 20ee$	13	$21ee = 372e - 1647$
$13 - 372e$	14	$21ee = 372e = - 1647$
$14 + \overline{21}$	15	$ee - \frac{124}{7}e = - \frac{549}{7}$
Comp. Square	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} =$
$16\,\mathit{lw}^{2}$	17	$e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$
$17 + \frac{62}{7}$	18	$e = \frac{62}{7} + \frac{1}{7} = 9$, or $8\frac{5}{7}$ the Mean
18×4	19	$4e = 36$, or $\frac{346}{7}$
8 and 19	20	$a = 36 - 31 = 5$, or $34\frac{6}{7} - 31 = 3\frac{6}{7}$
$18 + \overline{2}$	21	$2e = 18$, or $17\frac{3}{7}$
7 and 21	22	$y = 31 - 18 = 13$, or $31 - 17\frac{3}{7} = 13$

QUESTION VII.

350. There are three Numbers in Arithmetical Progreſ the Square of the 1ſt Term being added to the Product of th ther two, is 576; the Square of the mean being added to Product of the two Extreams makes 612, and the Squa the laſt Term being added to the Product of the firſt in ſecond is 792. What are thoſe Numbers?

SOLUTION.

Suppoſe	1	$a, e, y,$ in Arithmetical Progreſſion, as bef
Then	2	$aa + ye = 576$ by the Queſtion
	3	$ee + ya = 612$
	4	$yy + ae = 792$
$1 \because$	5	$a + y = 2e$

5 × e	6	$ae + ye = 2ee$	
2 + 4	7	$aa + ye + yy + ae = 1368$	
7 − 6	8	$aa + yy = 1368 - 2ee$	
− ee	9	$ya = 612 - ee$	
9 × 2	10	$2ya = 1224 - 2ee$	
+ 10	11	$aa + 2ya + yy = 2592 - 4ee$	
$5\,\odot^{2}$	12	$aa + 2ya + yy = 4ee$	
and 12	13	$4ee = 2592 - 4ee$	
+ 4ee	14	$8ee = 2592$	
4 ÷ 8	15	$ee = 324$	
$15\,l\omega^{2}$	16	$e = \sqrt{324} = 18$, the Mean	
.8,	17	$aa + yy = 1368 - 2ee = 720$	
10,	18	$2ya = 1224 - 2ee = 576$	
7 − 18	19	$aa - 2ya + yy = 720 - 576 = 144$	
$1\,l\omega^{2}$	20	$a - y = \sqrt{144} = 12$	
5 × 20	21	$2a = 2e + 12 = 48$	
21 ÷ 2	22	$a = 24$	Or, $\begin{cases} a = 12 \\ y = 24 \end{cases}$
5 − 22	23	$y = 20 - 24 = 12$	

Question VII.

351. It is required to find two ſuch Numbers, that their Sum ng ſubtracted from the Sum of their Squares, may leave 14, if their Product be added to their Sum it may make 14?

Solution.

Let a and e be put for the Numbers; and let $y = a + e$

Then $\{$	1	$aa + ee - y = 14$	$\}$ by the Queſtion.
	2	$ae + y = 14$	
1 + y	3	$aa + e = 14 + y$	
2 − y	4	$ae = 14 - y$	
4 × 2	5	$2ae = 28 - 2y$	
3 + 5	6	$aa + 2ae + ee = 42 - y$	
$6\,l\omega^{2}$	7	$a + e = \sqrt{42 - y}$	
But	8	$a + e = y$, as above	
7 and 8	9	$y = \sqrt{42 - y}$	
$9\,\odot^{2}$	10	$yy = 42 - y$	
10 + y	11	$yy + y = 42$	
compl.	12	$yy + y + \frac{1}{4} = 42 + \frac{1}{4} = 42{,}25$	
$12\,l\omega^{2}$	13	$y + \frac{1}{2} = \sqrt{42{,}25} = 6{,}5$	
$13 - \frac{1}{2}$	14	$y = 6{,}5 - \frac{1}{2} = 6$	

Conſequ.	15	$a + e = 6$, (for $a + e = y$) as above
3 and 14	16	$aa + ee = 14 + 6 = 20$
5 and 15	17	$2ae = 28 - 12 = 16$
16 — 17	18	$aa - 2ae + ee = 4$
8 ω^2	19	$a - e = \sqrt{4} = 2$
15 + 19	20	$2a = 8$
23 ÷ 2	21	$a = 4$
1 — 21	22	$e = 6 - 4 = 2$

QUESTION VIII.

352. Three Merchants join Stocks together; the firſt Man's Stock was leſs than the ſecond Man's 13*l.* the ſecond and third Man's Stock was 175*l.* In trading they gain 48*l.* more than their whole Stock was; the firſt Man's proportional Part of the Gain was 78*l.* What was each Man's Stock and Part of the Gain?

SOLUTION.

Let a, e, y, repreſent each Man's Stock.

Then	1	$a + e + y = s$, the whole Stock.
	2	$s + 48 =$ the whole Gain.
And	3	$a + 13 = e$
	4	$e + y = 175$
4 + a	5	$a + e + y = 175 + a$
1 and 5	6	$s = 175 + a$
6 and 2	7	$s + 48 = 223 + a$
But	8	$175 + a : 223 + a :: a : 78$, per Queſtion
8 ∵	9	$aa + 223a = 78a + 13650$
9 — 78 a	10	$aa + 145a = 13650$
	11	$aa + 145a + 5256{,}25 = 18906{,}25$
11 ω^2	12	$a + 72{,}5 = \sqrt{18906{,}25}$
12 — 72,5	13	$a = 1375 - 72{,}5 = 65$
3,	14	$e = a + 13 = 78$
4 — 14	15	$y = 97$
Then	16	65 : 78 :: 78 : 93*l.* 12*s.* e's Gain
Again	17	65 : 78 :: 97 : 116*l.* 8*s.* y's Gain
Proof	18	116*l.* 8*s.* + 93*l.* 12*s.* + 78*l.* = 288, the Gain
	19	65 + 78 + 97 = 240, the whole Stock
18 — 19	20	288 — 240 = 48, the Gain more than the Stock.

353. Theſe Examples will, for the preſent, be ſufficient for uadratic *Equations*, according to the beſt Methods of Soluti-n. What relates to adfected *Cubic*, and *higher Equations*, we hall defer 'till after we have proſecuted the Doctrine of *ſimple nd compound Intereſt*, which now the Reader is prepared for; nce if Equations, higher than *Quadratics*, occur in that Affair, hey are *ſimple ones*, and may be ſolved by *Logarithms*, or *Tables* hade for that Purpoſe.

CHAP. XVI.

f ANNUITIES, PENSIONS, &c. in Arrears, at SIMPLE INTEREST.

354. ANNUITIES, or Penſions, &c. are ſaid to be in Arrears, when they are payable or due, either yearly, r half-yearly, &c. and are unpaid for any Number of Payments. Therefore the Buſineſs is to compute what all thoſe ayments will amount unto, allowing any Rate of ſimple Intereſt for their Forbearance, for the Time each particular Payment became due: Now, in order to that,

$$\text{Put}\left\{\begin{array}{l}U = \text{the Annuity, Penſion, or yearly Rent, \&c.}\\ T = \text{the Time of its Continuance, or being unpaid.}\\ R = \text{the } \textit{Ratio}\text{, or Intereſt of 1}\textit{l.}\text{ for one Year, as before.}\\ A = \text{the Amount of the Annuity and its Intereſt.}\end{array}\right.$$

355. Then ſay, as 1*l.* : R :: U : RU, the Intereſt due at he End of the ſecond Year, for the Rent forborn one Year; nd 2 U will be the Rent or Annuity due at the ſame Time. o that for any given Time, the Intereſts and Rents will ſtand s below for each ſucceſſive Year, *viz.*

$$\left.\begin{array}{r}RU = \text{the Intereſt}\\ 2U = \text{the Rent}\end{array}\right\}\text{due at the End of the 2d Year.}$$

$$\left.\begin{array}{r}2RU = \text{the Intereſt}\\ 3U = \text{the Rent}\end{array}\right\}\text{due at the End of the 3d Year.}$$

$$\left.\begin{array}{r} 3RU = \text{the Intereſt} \\ 4U = \text{the Rent} \end{array}\right\} \text{due at the End of the 4th Year.}$$

$$\left.\begin{array}{r} 4RU = \text{the Intereſt} \\ 5U = \text{the Rent} \end{array}\right\} \text{due at the End of the 5th Year.}$$

And ſo on, for any Number of Years. Hence it is evident, that $RU + 2RU + 3RU + 4RU + 5RU = A$, the Sum of all the Rents and their Intereſt, being forborne five Years.

356. Let $T =$ the Time or Number of Years, then it follows that $RU + 2RU + 3RU + 4RU = A - TU$. Here $T = 5$; divide by U; then $R + 2R + 3R + 4R = \frac{A - TU}{U}$. Next to find the Sum of this Progreſſion: Thus, Let $R + 2R + 3R + 4R$, &c. $= Z$; then $1 + 2 + 3 + 4$, &c. $= \frac{Z}{R}$. Here the Sum of the firſt and laſt Terms are $4 + 1 = 5 = T$, and the Number of all the Terms is $4 = T - 1$: Therefore $\frac{T - 1}{2} \times T =$ the Sum of all the Terms; that is, $\frac{TT - T}{2} = \frac{Z}{R}$: Hence $\frac{TTR - TR}{2} = Z$, conſequently $\frac{TTR - TR}{2} = \frac{A - TU}{U}$.

What has been here ſaid may be eaſily deduced from Chap. 12, and Theorem in Article 318.

357. From the laſt Equation we raiſe the following Theorems. Theorem I. $\frac{TTU - TU}{2} \times R + TU = A$. Theorem II. $\frac{2A}{TTR - TR + 2T} = U$. Theorem III. $\frac{2A - 2TU}{TT - TU} = R$. (Let $\frac{2}{R} - 1 = x$.) Then Theorem IV. $T = \sqrt{\frac{2A}{RU} + \frac{xx}{4}} - \frac{1}{2}x$.

Question I.

358. If 250*l.* yearly Rent, (Penſion, &c.) be forborne or unpaid ſeven Years; what will it amount to in that Time, at 3 *per Cent.* for each Payment, as it becomes due?

Here

$$\text{Here is given}\left\{\begin{array}{l}U=250\\T=7\\R=0{,}03\end{array}\right\}\text{To find A, the Amount.}$$

$$\text{By Theorem I. } \frac{TTU-TU}{2}\times R+TU=\frac{7\times7\times250-7\times250}{2}$$

$$\times\,0{,}03+7\times250=\frac{10500}{2}\times0{,}03+7\times250=157{,}5$$

$$+\,1750=1907{,}5=1907\,l.\ 10\,s.=\text{the Amount required.}$$

359. If the Annuity or Penſion were to be paid *half-yearly*; then $U = 125$, $T = 14$, the Number of Payments, and $R = \frac{0{,}03}{2} = 0{,}015$, to find A; then the Theorem will be as follows:

$$TTU = 14\times14\times125 = 24500$$
$$TU = 14\times125 = 1750$$
$$TTU - TU = 24500 - 1750 = 22750$$
$$\frac{TTU-TU}{2}\times R = \frac{22750}{2}\times0{,}015 = 170{,}625$$

$170{,}625 + TU = 170{,}625 + 1750 = 1920{,}625 = A =$ 1920*l.* 12*s.* 6*d.*

Wherefore the half-yearly Payment is more advantageous than the yearly one by 13*l.* 2*s.* 6*d.* and conſequently quarterly Payments are ſtill more ſo than theſe.

QUESTION II.

360. What Annuity, Penſion, &c. being forborn, or unpaid ſeven Years, will raiſe a Stock of 1907*l.* 10*s.* at the Rate of 3 *per Cent.* Intereſt for each Payment as it becomes due?

$$\text{Here}\left\{\begin{array}{l}A=1907{,}5\\T=7\\R=6{,}03\end{array}\right\}\text{to find U, the Annuity, \&c.}$$

$$\text{Then per Theorem II. } \frac{2A}{TTR-TR+2T}=$$

$$\frac{2\times1907{,}5}{7\times7\times0{,}03-7\times0{,}03+2\times7}=250\,l.\ \text{the Annuity required.}$$

QUES-

QUESTION III.

361. If 250*l.* yearly Rent, being forborne ſeven Years, will amount to 1907*l.* 10*s.* allowing ſimple Intereſt for every Payment as it becomes due, what muſt the Rate of Intereſt be *per Cent*, &c.

$$\text{Here}\left\{\begin{array}{l}U = 250l.\\ A = 1907{,}5l.\\ T = 7 \text{ Years}\end{array}\right\}\text{to find R, the Rate of Intereſt.}$$

Per Theor. III. $\frac{2A-2TU}{TTU-TU}=R=\frac{2\times1907{,}5-2\times7\times250}{7\times7\times250-7\times250}$ $=0{,}03$; then $1 : 0{,}03 :: 100l. : 3l.$ the Intereſt *per Cent.* required.

QUESTION IV.

362. In what Time will 250*l.* yearly Rent, amount to 1907*l.* 10*s.* at 3*l.* *per Cent.* for Forbearance of Payment as they become due?

$$\text{Here}\left\{\begin{array}{l}U = 250l.\\ A = 1907{,}5\\ R = 0{,}03\end{array}\right\}\text{to find T, the Time, per Theor. II.}$$

Here $\frac{2}{R}-1=x=\frac{2}{0{,}03}-1=65{,}\cancel{6}$; and $\frac{1}{2}x=32{,}8\cancel{3}$; then $\sqrt{\frac{2A}{RU}+\frac{x^2}{4}}-\frac{1}{2}x=T=\sqrt{\frac{2\times1907{,}5}{0{,}03\times250}-\frac{65{,}\cancel{6}\times65{,}\cancel{6}}{4}}$ $-32{,}8\cancel{3}=7$, the Time, or Number of Years required.

CHAP. XVII.

The PRESENT WORTH *of* ANNUITIES, PENSIONS, *&c. computed at* SIMPLE INTEREST.

363. THE Buſineſs of purchaſing Annuities, or taking of Leaſes, &c. for any aſſigned Time, depends upon the true equating of the Principal, or Money, laid out on the

Purchaſe, with the Annuity or yearly Rent, by allowing (or diſcounting) the ſame Rate of Intereſt to both Parties; which may be eaſily performed, by duly applying the reſpective Theorems in Article 270 and 357, for finding the Amount of any Annuity and its Intereſt, forborne for any given Time; and then what Sum or Principal, put to Intereſt at the ſame Rate, will in the ſame Time amount to the ſame Sum.

364. Now ſince the Amount A in both Caſes is the ſame, the Theorems which give the ſaid Amount, muſt be equal to each other; that is, (Article 357. Theorem I.) $\frac{ttU - tU}{2} \times R + tU = tPR + P = A$. (Article 270.) Which two Theorems compounded together give the following, *viz.*

$$\text{Theorem I. } \frac{ttR - tR + 2t}{2tR + 2} \times U = P.$$

$$\text{Theorem II. } \frac{tR + 1}{ttR - tR + 2t} \times 2P = U$$

$$\text{Theorem III. } \frac{2P - 2tU}{ttU - tU - 2Pt} = R$$

Now put $\frac{2}{R} - \frac{2P}{U} - 1 = x$; then $tt \pm xt = \frac{2P}{RU}$, which gives

$$\text{Theorem IV. } t = \sqrt{\frac{2P}{RU} + \frac{xx}{4}} \pm \frac{x}{2}$$

QUESTION I.

365. What is 75*l.* yearly Rent, to continue nine Years, worth in Ready Money, at 3 *per Cent.*

$$\text{Here}\left\{\begin{array}{l} U = 75l. \\ R = 0{,}03 \\ t = 9 \text{ Years} \end{array}\right\}\text{to find the preſent Worth P.}$$

Theor. I. $\frac{ttR - tR + 2t}{2tR + 2} \times U = \frac{9 \times 9 \times 0{,}03 - 9 \times 0{,}03 + 2 \times 9}{2 \times 9 \times 0{,}03 + 2} \times 75 = \frac{2{,}43 - 0{,}27 + 18}{0{,}54 + 2} \times 75 = 611l.\ 4s.\ 4d.\frac{3}{4}$ the preſent Worth required.

QUES-

QUESTION II.

366. What Annuity, to continue 21 Years, will 192*l.* 5½*d.* purchaſe, at 5 *per Cent?*

$$\text{Here}\left\{\begin{array}{l} P = 192{,}0731\,l. \\ t = 21 \text{ Years} \\ R = 0{,}05 \end{array}\right\}\text{to find the Annuity U.}$$

Then, Theor. II. in Numbers, $\dfrac{21 \times 0{,}05 + 1}{21\times21\times0{,}05 - 21\times0{,}05 + 2\times2}$ $\times\, 2 \times 192{,}0731 = 12{,}5 = 12\,l.\ 10\,s.$ the Annuity required.

QUESTION III.

367. At what Rate of ſimple Intereſt will 192*l.* 1*s.* purchaſe an Annuity of 12*l.* 10*s.* to continue 21 Years?

$$\text{Here}\left\{\begin{array}{l} P = 192{,}0731\,l. \\ U = 12{,}5\,l. \\ T = 21 \text{ Years} \end{array}\right\}\text{to find R.}$$

Per Theor. III. $\dfrac{2 \times 192{,}0731 - 2 \times 21 \times 12{,}5}{21\times21\times12{,}5 - 21\times12{,}5 - 2\times21\times192{,}07}$ $= 0{,}05$; then 1 : 0,05 :: 100*l.* : 5*l.* the Intereſt *per Cent.* lowed.

QUESTION IV.

368. In what Time will 7*l.* *per Ann.* pay a Debt of 120*l.* at 6*l.* *per Cent.* Intereſt? Or thus; For how long a may an Annuity of 7*l.* *per Ann.* be purchaſed, or enjoyed, 120*l.* 8*s.* at the aforeſaid Rate of Intereſt?

$$\text{Here}\left\{\begin{array}{l} U = 7\,l. \\ R = 0{,}06 \\ P = 120{,}4\,l. \end{array}\right\}\text{to find T, the Time.}$$

Then $\dfrac{2}{R} - \dfrac{2P}{U} + 1 = x = 2{,}0\cancel{6}$. (See Article 364.)

$\sqrt{\dfrac{2P}{RU} + \dfrac{xx}{4}} + \tfrac{1}{2}x = T = 24{,}\cancel{9} = 25$ Years, the required.

CHA

CHAP. XVIII.

e *Conſtruction and Uſe of* TABLES *of* SIMPLE INTEREST.

9. THE great Deſign of Tables of Intereſt (both Simple
and Compound) is Eaſe and Expedition in practical
lculations. For, beſides that the Rules expreſſed in Words
anſwering Queſtions of Intereſt are tedious and intricate,
l the Reaſon no Ways to be underſtood, the Operations them-
es are, for the moſt Part, very laborious; and conſequently
bles which expedite and facilitate the Practice are indiſpen-
y neceſſary.

370. Theſe Tables are made in Decimal Numbers, the firſt
Days, and the ſecond for Years; which Numbers, being
ithmetical Proportion, make them capable of that Perfection
ich no other Tables can pretend to: They are ſo contrived,
t the Intereſt of any principal Sum is eaſily found for any
mber of Days, or Years, at any Rate, from one Pound to
, with the Halves and Quarters; having followed herein the
v. Mr. *Brown* in his *Arithmetica Infinita.*

371. The Conſtruction of theſe Tables is eaſy from the The-
ms themſelves, (and indeed the Reaſon of their Conſtruction
be no otherways ſo eaſily conceived.) Thus, by Theorem
of Simple Intereſt, *viz.* $tRP + P = A$ is the firſt and ſecond
ble conſtructed. For ſince the Amount leſs the Principal,
qual to the Intereſt, therefore the Theorem will be tRP
Intereſt. Now if $P = 1$*l.* $t = ,002739$, &c. (the Deci-
l of a Year for one Day) and $R =$ any Ratio of Intereſt; ſup-
e 5 *per Cent.* then the Simple Intereſt of one Pound for one
y, at 5 *per Cent.* is $,002739$, &c. $\times ,05 \times 1 = 00013698$,
. which being multiplied by the nine Digits, ſeverally con-
ute that Part of the Table of Intereſt at 5 *per Cent.* and
s the whole firſt Table is made. The ſecond Table, for
ars, is only the various *Ratios* of Intereſt multiplied by the
l nine Digits; for ſince $t = 1$ Year, and $P = 1$*l.* it will be
$P = R$, the Intereſt for the firſt Year, &c.

372. The third Table ſhews the Rebate or Diſcount to be made for one Pound, at the ſeveral Rates *per Cent.* for Days. The Manner, Truth, and Reaſon of its Conſtruction is derived from Theorem II. of Simple Intereſt, *viz.* $\frac{A}{tR+1}=P.$ For ſince the Principal or preſent Worth, ſubducted from the Amount, gives the Rebate or Diſcount of that Amount; therefore the Diſcount of any Amount for any Time, at any Rate, (without Regard of the preſent Value, or principal Money) may be found by this Theorem $\frac{AtR}{tR+1}=D=$ Diſcount. $\left(\text{For } A-\frac{A}{tR+1}=\frac{AtR}{tR+1}.\right)$ Hence if we put $A=1$ *l.* $t=,002739$, *&c.* and $R=$ any *Ratio* of Intereſt, ſuppoſe 5 *per Cent.* then by this laſt Theorem we have the Diſcount of one Pound for one Day, at the Rate of 5 *per Cent. per Ann.* for $AtR=1\times,002739$, *&c.* $\times,05=,00013698$, *&c.* And $tR+1=1,00013698$, *&c.* then by Diviſion, 1,00013698, *&c.*) ,00013698, *&c.* ($=,00013697$, *&c.* the Diſcount. If $t=1$ Year, then the annual Diſcount of one Pound at 5 *per Cent.* will be found, by the above Theorem, thus; $AtR=,05$, and $tR+1=1,05$. Therefore by Diviſion, 1,05) ,05 ($=,04761904$, *&c.* the Diſcount. And thus is the Diſcount of any Sum, at any Rate for any Time above one Year, found at once by the above Theorem; and for any Time under a Year by the Table of Diſcount for Days, of which I have now taught the Conſtruction in a new and more rational Method than any I have yet ſeen.

The Uſe of Table I. *and* II.

373. In order to underſtand how to make thoſe two Tables univerſally uſeful, the Reader is to obſerve, that if a Number conſiſts of only one Digit, with Cyphers affixed, as 10, 50, 700, 9000, 800000, *&c.* it is called a *pure Number*; but thoſe Numbers which conſiſt of more than one, or wholly of Digits, as 370, 568, 7569, *&c.* may be called *mixed Numbers.* Now every mixed Number may be reſolved into thoſe pure Numbers, of which they are compoſed; thus the mixed Number 567 may

be reſolved into the pure Numbers 500, 60, and 7; ſo alſo 15890 is reſolved into 10000, 5000, 800, and 90.

374. Then as to the Uſe of the Tables, obſerve theſe Rules:

I. If the Number of Days, Years, *&c.* propoſed, be a mixed Number, let it be reſolved into pure Numbers.

II. With the pure Numbers ſeverally enter the Tables, and take thoſe Decimal Numbers which ſtand againſt the firſt Figure of each pure Number, in the Column marked *Numbers.*

III. Remove the Decimal Point in each ſuch Decimal Number, ſo many Places to the Right Hand, as there are Cyphers in the reſpective pure Numbers.

IV. Laſtly, Add together all the Decimal Numbers, and find the Value thereof by the Tables for that Purpoſe.

Theſe Things premiſed, the Uſe of the Tables will be obvious from the Examples of the following Problems.

PROBLEM I.

375. To find the Intereſt of any Sum of Money for a Day, or a Year, at any Rate *per Cent. per Ann.*

EXAMPLE I.

What is the Intereſt of 2746*l.* at 5*l.* 15*s.* *per Cent.* for a Day?

Decimals.

$$\left.\begin{array}{l}\text{In Table I. under the}\\ \text{Rate } 5\tfrac{3}{4}\text{, you find a-}\\ \text{gainſt the pure Num-}\\ \text{bers}\end{array}\right\}\begin{array}{r}2000 \text{ — } ,31506\\ 700 \text{ — } ,11027\\ 40 \text{ — } ,00630\\ 6 \text{ — } ,00094\end{array}$$

The Anſwer is — ,43257 = 8*s.* $7\frac{3}{4}$*d.*

EXAMPLE II.

376. What is the Intereſt of the ſame Sum, at the ſame Rate for a Year?

		Decimals.	
In Table II. you find againſt	2000 —	115,00000	Under 5¾*d.* *per Cent.*
	700 —	40,25000	
	40 —	2,30000	
	6 —	0,34500	

The Anſwer in Decimals *l.* 157,895

Which is in Money = 157*l.* 17*s.* 10¾*d.*

PROBLEM II.

377. To find the Intereſt of any Sum of Money for a Number of Days.

EXAMPLE.

What is the Intereſt of 265*l.* for 149 Days, at the Rate o 3*l.* 15*s.* *per Cent.* &c.?

Multiply the principal Sum	—	265*l.*
By the given Number of Days	—	149

The Product is the mixed Number 39485, with whic reſolved, enter the Table as before:

		Decimals.	
Thus in Table I. you find againſt the pure Numbers	30000 —	3,08220	Under 3¾*d.* *per Cent.*
	9000 —	0,92466	
	400 —	0,04109	
	80 —	0,00822	
	5 —	0,00051	

The Anſwer in Decimals *l.* 4,05668

In Money 4*l.* 1*s.* 1½*d.*

The Method is the ſame for any greater Number of Days.

PROBLEM III.

378. To find the Intereſt of any Sum forborne any Numbe of Years, at any of the given Rates *per Cent.*

EXAMPLE.

What is the Intereſt of 175*l.* 15*s.* forborne 13 Years, at the Rate of 6 *per Cent.* &c.?

Multiply the principal Sum — 175,75
By the Number of Years given — 13

The Product is the mixed Number 2284,75

Which reſolved, as before, will ſtand thus,

In Table II. you find againſt	2000, —	120,000	Under 6 *per Cent.*
	200, —	12,000	
	80, —	4,800	
	4, —	0,240	
	,7 —	0,042	
	,05 —	0,003	

The Anſwer in Decimals *l.* 137,085

The ſame in Money 137*l.* 1*s.* 8$\frac{1}{4}$*d.*

N. B. The Reader muſt obſerve, in reſolving a mixed Number, wherein are Decimals, to remove the Point one Place more to the *left*, than are the Number of Cyphers in the Decimal pure Number, as in the laſt Example.

The Uſe of Table III. *Of* Diſcount.

379. In ſeeking the Diſcount for any Sum due at the End of any Number of Days, if the Number of Days be a mixed one, reſolve them into pure Numbers as before taught; and even with them in the Table take the Diſcount of 1*l.* which add together; and then multiply by the principal Sum, the Product will be the Diſcount thereof.

PROBLEM IV.

To find the Diſcount of any Sum, for any Number of Days, at any given Rate in the Table.

EXAMPLE.

What is the Diſcount of 83*l.* 10*s.* for 235 Days, at 4 *per Cent. per Annum?*

Deci-

			Decimals.	
You find in the Table even with	200	—	,0214478	Under 4 *per* Cent.
	30	—	,0032769	
	5	—	,0005476	
The Sum is	—	—	,0252723	
Which multiplied by the Sum			83,5	
				l. *s.* *d.*
The Product is the Anſwer			2,110237	= 2 — 2 — 2¼

Problem V.

380. To find the Diſcount of any Sum for a Year.

Example.

What is the Diſcount of 100*l.* for one Year, at 5 *per Cent?*

		l. *s.* *d.*
In the Table under 5 *per Cent.* and againſt 365 Days, is	,047619, &c.	
Which multiplied by the principal Sum ——	100	
The Product is the Anſwer	*l.* 4,7619, &c.	= 4 — 15 — 2¾
Now the Intereſt of 100*l.* for one Year, at 5 *per Cent.* is — — — —		5 — 0 — 0
The Difference therefore of Diſcount and Intereſt is — — — —		0 — 4 — 9¼

Whence it is evident, he who allows Intereſt for Diſcount wrongs himſelf conſiderably, which yet is very common among Traders; for ſo much Money ought to be paid as, at Intereſt, would amount to the Sum due, in the Time propoſed.

Example II.

381. What is the Diſcount of 9342*l.* at 4½ *per Cent.* for a Year?

The

The Diſcount of 1*l.* for 365 Days, at 4½ *per Cent.* in the Table, is	,043062, *&c.*
Which multiplied by the principal Sum	9342
The Product is the Anſwer —	*l.* 402,2852, *&c.*

In Money 402*l.* 5*s.* 8½*d.*

And thus proceed for other annual Diſcounts.

It muſt be acknowledged this Table of Diſcount gives not the preciſe Truth, and yet differs but little from it; being ſufficiently exact for any Uſe. None but a Table of the Diſcount for every Day can be perfect; becauſe every Day's Diſcount differs, being ſtill leſs as the Number of Days increaſe.

This Table is perfectly true for all the Days expreſſed therein, and, as I ſaid, may be uſed without much Error for any other.

		l.	*s.*	*d.*
Example in Prob. IV.	The true Diſcount is —	2	1	11
	The Diſcount by this Table	2	2	2¼
	The Intereſt for the Time and Rate	2	3	0

TABLES

TABLES

OF

SIMPLE INTEREST.

TABLE I.

The Interest of one Pound per Diem, *at any Rate* per Cent. *from one to ten Pounds, with Halves and Quarters.*

Numb.	1 *per Cent.*	1¼ *per Cent.*	1½ *per Cent.*	1¾ *per Cent.*
1	,00002740	,00003425	,00004110	,00004794
2	,00005480	,00006850	,00008220	,00009589
3	,00008220	,00010274	,00012329	,00014383
4	,00010959	,00013699	,00016438	,00019178
5	,00013698	,00017123	,00020548	,00023972
6	,00016438	,00020548	,00024657	,00028767
7	,00019178	,00023973	,00028767	,00033562
8	,00021918	,00027398	,00032877	,00038356
9	,00024657	,00030822	,00036986	,00043151
Month.		,00104166	,00125000	,00145833

Numb.	2 *per Cent.*	2¼ *per Cent.*	2½ *per Cent.*	2¾ *per Cent.*
1	,00005480	,00006164	,00006849	,00007534
2	,00010959	,00012329	,00013699	,00015068
3	,00016438	,00018493	,00020547	,00022602
4	,00021918	,00024657	,00027397	,00030137
5	,00027397	,00030822	,00034246	,00037671
6	,00032876	,00036986	,00041095	,00045205
7	,00038356	,00043151	,00047945	,00052739
8	,00043835	,00049315	,00054794	,00060274
9	,00049315	,00055479	,00061644	,00067808
Month.	,00166666	,00187500	,00208333	,00229166

Numb.

Numb.	3 *per Cent.*	3¼ *perCent.*	3½ *perCent.*	3¾ *perCent.*
1	,00008220	,00008904	,00009589	,00010274
2	,00016438	,00017808	.00019178	,00020548
3	,00024657	.00026712	,00028767	,00030822
4	,00032877	,00035616	,00038356	,00041096
5	,00041096	,00044520	,00047945	,00051363
6	,00049315	,00053424	,00057534	,00061644
7	,00057534	,00062328	,00067123	,00071917
8	,00065753	,00071232	,00076712	,00082192
9	,00073972	,00080137	,00086301	,00092465
Month.	,00250000	,00270833	,00291666	,00312500

Days.	4 *per Cent.*	4¼ *perCent.*	4½ *perCent.*	4¾ *perCent.*
1	,00010959	,00011644	,00012329	,00013014
2	,00021918	,00023288	,00024657	,00026027
3	,00032877	,00034931	,00036986	,00039041
4	,00043836	,00046575	,00049315	,00052055
5	,00054794	,00058219	,00061643	,00065068
6	,00065753	,00069863	,00073973	,00078082
7	,00076712	,00081507	,00086301	,00091096
8	,00087671	,00093151	,00098630	,00104109
9	,00098630	,00104794	,00110959	,00117123
Month.	,00333333	,00354166	,00375000	,00395833

Days.	5 *per Cent*	5¼ *perCent.*	5½ *perCent.*	5¾ *per Cent.*
1	,00013698	,00014383	,00015068	,00015753
2	,00027397	,00028767	,00030137	,00031507
3	,00041096	,00043151	,00045205	,00047260
4	,00054794	,00057534	,00060274	,00063014
5	,00068493	,00071918	,00075342	,00078767
6	,00082192	,00086301	,00090411	,00094520
7	,00095890	,00100685	,00105479	,00110274
8	,00109589	,00115068	,00120548	,00216027
9	,00123288	,00129452	,00135616	.00141781
Month.	,00416666	,00437500	,00458333	'00479166

Days.	6 *per Cent.*	6¼ *perCent.*	6½ *perCent.*	6¾ *perCent.*
1	.00016438	.00017123	.00017808	.00018493
2	.00032876	.00034246	.00035616	.00036986
3	.00049315	.00051370	.00053424	.00055479
4	.00065753	.00068493	.00071232	.00073972
5	.00082192	.00085616	.00089041	.00092466
6	.00098630	.00102740	.00106849	.00110959
7	.00115068	.00119863	.00124657	.00129452
8	.00131507	.00136986	.00142465	.00147945
9	.00147945	.00154109	.00160274	.00166438
Month.	.00500000	.00520833	.00541666	.00562500

Days.	7 *per Cent.*	7¼ *per Cent.*	7½ *perCent.*	7¾ *perCent.*
1	.00019178	.00019863	.00020548	.00021233
2	.00038356	.00039726	.00041096	.00042466
3	.00057534	.00059589	.00061644	.00063699
4	.00076712	.00079452	.00082192	.00084932
5	.00095890	.00099315	.00102739	.00106164
6	.00115068	.00119178	.00123288	.00127397
7	.00134246	.00139041	.00143836	.00148630
8	.00153425	.00158904	.00164384	.00169863
9	.00172603	.00178767	.00184932	.00191096
Month.	.00583333	.00604166	.00625000	.00645833

Days.	8 *per Cent.*	8¼ *perCent.*	8½ *perCent.*	8¾ *perCent.*
1	.00021918	.00022603	.00023287	.00023973
2	.00043835	.00045205	.00046575	.00047945
3	.00065753	.00067808	.00069863	.00071918
4	.00087671	.00090411	.00093150	.00095890
5	.00109589	.00113014	.00116438	.00119863
6	.00131507	.00135616	.00139726	.00143835
7	.00153425	.00158219	.00163013	.00167808
8	.00175342	.00180822	.00186301	.00191781
9	.00197260	.00203424	.00209589	.00215753
Month.	.00666666	.00687500	.00708333	.00729166

Days.	9 *per Cent.*	9¼ *perCent.*	9½ *perCent.*	9¾ *perCent.*
1	.00024657	.00025342	.00026028	.00026712
2	.00049315	.00050684	.00052055	.00052424
3	.00073972	.00076027	.00078082	.00080137
4	.00098630	.00101370	.00104109	.00106849
5	.00123287	.00126712	.00130137	.00133561
6	.00147945	.00152955	.00156164	.00160274
7	.00172602	.00177397	.00182192	.00186986
8	.00197260	.00202739	.00208219	.00213699
9	.00221918	.00228082	.00234246	.00240410
Month.	.00750000	.00770833	.00791666	.00812500

TABLE II.

3. *The Intereſt of one Pound* per Annum, *at any Rate* per Cent. *from one to ten Pounds, with Halves and Quarters.*

Numb.	1 *per Cent.*	1¼ *per Cent.*	1½ *per Cent.*	1¾ *per Cent.*
1	0.01000000	0.01250000	0.01500000	0.01750000
2	0.02000000	0.02500000	0.03000000	0.03500000
3	0.03000000	0.03750000	0.04500000	0.05250000
4	0.04000000	0.05000000	0.06000000	0.07000000
5	0.05000000	0.06250000	0.07500000	0.08750000
6	0.06000000	0.07500000	0.09000000	0.10500000
7	0.07000000	0.08750000	0.10500000	0.12250000
8	0.08000000	0.10000000	0.12000000	0.14000000
9	0.09000000	0.11250000	0.13500000	0.15750000

Numb.	2 *per Cent.*	2¼ *per Cent.*	2½ *per Cent.*	2¾ *per Cent.*
1	0.02000000	0.02250000	0.02500000	0.02750000
2	0.04000000	0.04500000	0.05000000	0.05500000
3	0.06000000	0.06750000	0.07500000	0.08250000
4	0.08000000	0.09000000	0.10000000	0.11000000
5	0.10000000	0.11250000	0.12500000	0.13750000
6	0.12000000	0.13500000	0.15000000	0.16500000
7	0.14000000	0.15750000	0.17500000	0.19250000
8	0.16000000	0.18000000	0.20000000	0.22000000
9	0.18000000	0.20250000	0.22500000	0.24750000

Numb.	3 *per Cent.*	3¼ *per Cent.*	3½ *per Cent.*	3¾ *per Cent.*
1	0.03000000	0.03250000	0.03500000	0.03750000
2	0.06000000	0.06500000	0.07000000	0.07500000
3	0.09000000	0.09750000	0.10500000	0.11250000
4	0.12000000	0.13000000	0.14000000	0.15000000
5	0.15000000	0.16250000	0.17500000	0.18750000
6	0.18000000	0.19500000	0.21000000	0.22500000
7	0.21000000	0.22750000	0.24500000	0.26250000
8	0.24000000	0.26000000	0.28000000	0.30000000
9	0.27000000	0.29000000	0.31500000	0.33750000

Years.	4 *per Cent.*	4¼ *per Cent.*	4½ *per Cent.*	4¾ *per Cent.*
1	0.04000000	0.04250000	0.04500000	0.04750000
2	0.08000000	0.08500000	0.09000000	0.09500000
3	0.12000000	0.12750000	0.13500000	0.14250000
4	0.16000000	0.17000000	0.18000000	0.19000000
5	0.20000000	0.21250000	0.22500000	0.23750000
6	0.24000000	0.25500000	0.27000000	0.28500000
7	0.28000000	0.29750000	0.31500000	0.33250000
8	0.32000000	0.34000000	0.36000000	0.38000000
9	0.36000000	0.38250000	0.40500000	0.42750000

Years.	5 *per Cent.*	5¼ *per Cent.*	5½ *per Cent.*	5¾ *per Cent.*
1	0.05000000	0.05250000	0.05500000	0.05750000
2	0.10000000	0.10500000	0.11000000	0.11500000
3	0.15000000	0.15750000	0.16500000	0.17250000
4	0.20000000	0.21000000	0.22000000	0.23000000
5	0.25000000	0.26250000	0.27500000	0.28750000
6	0.30000000	0.31500000	0.33000000	0.34500000
7	0.35000000	0.36750000	0.38500000	0.40250000
8	0.40000000	0.42000000	0.44000000	0.46000000
9	0.45000000	0.47250000	0.49500000	0.51750000

Years.	6 *per Cent.*	6¼ *per Cent.*	6½ *per Cent.*	6¾ *per Cent.*
1	0.06000000	0.06250000	0.06500000	0.06750000
2	0.12000000	0.12500000	0.13000000	0.13500000
3	0.18000000	0.18750000	0.19500000	0.20250000
4	0.24000000	0.25000300	0.26000000	0.27000000
5	0.30000000	0.31250000	0.32500000	0.33750000
6	0.36000000	0.37500000	0.39000000	0.40550000
7	0.42000000	0.43750000	0.45500000	0.47250000
8	0.48000000	0.50000000	0.52000000	0.54000000
9	0.54000000	0.56250000	0.58500000	0.60750000

Years.	7 *per Cent.*	7¼ *per Cent.*	7½ *per Cent.*	7¾ *per Cent.*
1	0.07000000	0.07250000	0.07500000	0.07750000
2	0.14000000	0.14500000	0.15000000	0.15500000
3	0.21000000	0.21750000	0.22500000	0.23250000
4	0.28000000	0.29000000	0.30000000	0.31000000
5	0.35000000	0.36250000	0.37500000	0.38750000
6	0.42000000	0.43500000	0.45000000	0.46500000
7	0.49000000	0.50750000	0.52500000	0.54250000
8	0.56000000	0.58000000	0.60000000	0.62000000
9	0.63000000	0.65250000	0.67500000	0.69750000

Years.	8 *per Cent.*	8¼ *per Cent.*	8½ *per Cent.*	8¾ *per Cent.*
1	0.08000000	0.08250000	0.08500000	0.08750000
2	0.16000000	0.16500000	0.17000000	0.17500000
3	0.24000000	0.24750000	0.25500000	0.26250000
4	0.32000000	0.33000000	0.34000000	0.35000000
5	0.40000000	0.41250000	0.42500000	0.43750000
6	0.48000000	0.49500000	0.51000000	0.52500000
7	0.56000000	0.57750000	0.59500000	0.61250000
8	0.64000000	0.66000000	0.68000000	0.70000000
9	0.72000000	0.74250000	0.76500000	0.78750000

Years.	9 *per Cent.*	9¼ *per Cent.*	9½ *per Cent.*	9¾ *per Cent.*
1	0.09000000	0.09250000	0.09500000	0.09750000
2	0.18000000	0.18500000	0.19000000	0.19500000
3	0.27000000	0.27750000	0.28500000	0.29250000
4	0.36000000	0.37000000	0.38000000	0.39000000
5	0.45000000	0.46250000	0.47500000	0.48750000
6	0.54000000	0.55500000	0.57000000	0.58500000
7	0.63000000	0.64750000	0.66500000	0.68250000
8	0.72000000	0.74000000	0.76000000	0.78000000
9	0.81000000	0.83250000	0.85500000	0.87750000

TABLE III.

Of SIMPLE INTEREST.

384. *The* Rebate, *or* Diſcount *of one Pound for* Days, *at the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, 6, per Cent. per Annum.

Days.	*2 per Cent.*	*2½ per Cent.*	*3 per Cent.*	*3½ per Cent.*
1	.0000548	.0000685	.0000822	.0000959
2	.0001096	.0001370	.0001644	.0001917
3	.0001644	.0002054	.0002465	.0002876
4	.0002191	.0002739	.0003287	.0003834
5	.0002739	.0003424	.0004108	.0004792
6	.0003287	.0004108	.0004929	.0005750
7	.0003834	.0004792	.0005750	.0006708
8	.0004382	.0005477	.0006571	.0007666
9	.0004929	.0006161	.0007392	.0008623
10	.0005477	.0006845	.0008212	.0009580
20	.0010947	.0013680	.0016411	.0019141
30	.0016411	.0020506	.0024597	.0028685
40	.0021870	.0027322	.0032769	.0038210
50	.0027322	.0034139	.0040928	.0047716
60	.0032769	.0040928	.0049073	.0057205
70	.0038210	.0047716	.0057205	.0066676
80	.0043644	.0054496	.0065324	.0076128
90	.0049073	.0061266	.0073429	.0085563
100	.0054496	.0068027	.0081522	.0094980
110	.0059913	.0074779	.0089601	.0104379
120	.0065324	.0081522	.0097667	.0113760
130	.0070729	.0088255	.0105720	.0123123
140	.0076128	.0094980	.0113760	.0132468
150	.0081522	.0101695	.0121786	.0141796
160	.0086909	.0108401	.0129780	.0151106
170	.0092291	.0115098	.0137801	.0160399
180	.0097667	.0121786	.0145788	.0169674
190	.0103037	.0128465	.0153763	.0178932
200	.0108401	.0135135	.0161725	.0188172
210	.0113759	.0141796	.0169674	.0197395

TABLE III.

The Diſcount of one Pound for Days:

Days.	4 *per Cent.*	4½ *per Cent.*	5 *per Cent.*	6 *per Cent.*
1	.0001096	.0001233	.0001370	.0001644
2	.0002191	.0002465	.0002739	.0003287
3	.0003287	.0003697	.0004108	.0004929
4	.0004382	.0004929	.0005477	.0006571
5	.0005477	.0006161	.0006845	.0008212
6	.0006571	.0007392	.0008212	.0009853
7	.0007665	.0008623	.0009580	.0011494
8	.0008759	.0009853	.0010947	.0013133
9	.0009853	.0011084	.0012314	.0014773
10	.0010947	.0012314	.0013680	.0016411
20	.0021870	.0024597	.0027322	.0032769
30	.0032769	.0036850	.0040928	.0049073
40	.0043644	.0049073	.0054496	.0065324
50	.0054496	.0061266	.0068027	.0081522
60	.0065234	.0073429	.0081522	.0097667
70	.0076128	.0085563	.0094980	.0113760
80	.0086909	.0097667	.0108401	.0129780
90	.0097667	.0109741	.0121786	.0145788
100	.0108401	.0121786	.0135135	.0161725
110	.0119112	.0133802	.0148448	.0177610
120	.0129800	.0145788	.0161725	.0193444
130	.0140465	.0157746	.0174966	.0209228
140	.0151006	.0169674	.0188172	.0224960
150	.0161725	.0181574	.0201342	.0240642
160	.0172321	.0193444	.0214477	.0256273
170	.0182894	.0205286	.0227577	.0271855
180	.0193444	.0217100	.0240642	.0287387
190	.0203972	.0228885	.0253672	.0302869
200	.0214477	.0240642	.0266667	.0318302
210	.0224960	.0252370	.0279627	.0333686

TABLE III.

The Discount of one Pound for Days.

Days.	*2 per Cent.*	*2 ½ per Cent.*	*3 per Cent.*	*3 ½ per Cent.*
220	.0119112	.0148448	.0177610	.0206601
230	.0124459	.0155091	.0185534	.0215789
240	.0129800	.0161725	.0193444	.0224959
250	.0135135	.0168350	.0201342	.0234114
260	.0140465	.0174966	.0209227	.0243251
270	.0145788	.0181574	.0217100	.0252370
280	.0151106	.0188172	.0224960	.0261473
290	.0156418	.0194762	.0232807	.0270558
300	.0161725	.0201342	.0240642	.0279627
310	.0167026	.0207914	.0248464	.0288679
320	.0172321	.0214477	.0256273	.0297714
330	.0177610	.0221031	.0264070	.0306732
340	.0182894	.0227577	.0271855	.0315734
350	.0188172	.0234114	.0279627	.0324718
360	.0193444	.0240642	.0287387	.0333686
361	.0193971	.0241294	.0288162	.0334582
362	.0194498	.0241946	.0288937	.0335478
363	.0195025	.0242598	.0289712	.0336374
364	.0195552	.0243251	.0290487	.0337269
365	.0196078	.0243902	.0291262	.0338164

TABLE III.

The Diſcount of one Pound for Days.

Days.	4 *per Cent.*	4½ *per Cent.*	5 *per Cent.*	6 *per Cent.*
220	.0235420	.0264070	.0292553	.0349022
230	.0245858	.0275743	.0305445	.0364309
240	.0256273	.0287387	.0318302	.0379547
250	.0266667	.0299003	.0331126	.0394737
260	.0277038	.0310592	.0343915	.0409879
270	.0287387	.0322153	.0356671	.0424974
280	.0297714	.0333686	.0369393	.0440021
290	.0308019	.0345192	.0382082	.0455021
300	.0318302	.0356671	.0394737	.0469974
310	.0328564	.0368122	.0407352	.0484880
320	.0338804	.0379547	.0419948	.0499740
330	.0349022	.0390444	.0432503	.0514553
340	.0359218	.0402314	.0445026	.0529320
350	.0369393	.0413657	.0457516	.0544041
360	.0379547	.0424974	.0469974	.0558717
361	.0380561	.0426104	.0471218	.0560182
362	.0381575	.0427234	.0472462	.0561647
363	.0382588	.0428364	.0473705	.0563111
364	.0383602	.0429493	.0474948	.0564575
365	.0384615	.0430622	.0476191	.0566038

CHAP. XIX.

Of Compound Intereſt, Annuities, &c.

5. Compound Intereſt is that which ariſes from any Principal and its Intereſt put together, as the Intereſt ſo be- mes due, ſo that at every Payment, or at the Time when the ayments became due, there is created a new Principal; and r that Reaſon it is called Intereſt upon Intereſt, or Compound tereſt. As for Example: Suppoſe 100*l.* were lent out for o Years at 6*l.* *per Cent. per Ann.* Compound Intereſt; then the End of the firſt Year it will only amount to 106*l.* as in mple Intereſt; but for the ſecond Year this 106*l.* becomes rincipal, which will amount to 112*l.* 7*s.* 2½*d.* at the ſecond ears End, whereas by Simple Intereſt would have amounted but 112*l.* And though it be not lawful to let out Money at ompound Intereſt, yet in purchaſing of Annuities or Penſions, c. and taking Leaſes in Reverſion, it is very uſual to allow ompound Intereſt to the Purchaſer for his ready Money; and erefore it is requiſite to underſtand it.

386. Now in order to raiſe the Theorems for Compound Inreſt, we muſt conſider that the Amount of 1*l.* and its Inteſt, for one Year, is here called the *Ratio*, and is found by the llowing Proportion for any Rate of Intereſt. Thus,

$$\text{As}\left\{\begin{array}{l} \quad\pounds. \\ 100 : 105 :: 1 : 1{,}05 = \text{R, at } 5 \textit{ per Cent.} \\ 100 : 106 :: 1 : 1{,}06 = \text{R, at } 6 \textit{ per Cent, } \&\text{c.} \end{array}\right.$$

387. But as one Pound is to the Amount of one Pound at e Year's End, ſo is that Amount to the Amount of one ound at two Years End; and ſo on continually.

That is, $1 : R :: R : R :: RR : RRR :: R^3 : R^4 :: R^4 ::$
$^5 ::$ &c.

$$\text{Then}\left\{\begin{array}{l} 1.\ \ 2.\ \ 3.\ \ 4.\ \ 5. = \text{Years.} \\ R.\ R^2.\ R^3.\ R^4.\ R^5 = \text{the Amount of } 1l. \text{ at any} \\ \text{Rate.} \end{array}\right.$$

388. Hence 'tis evident the Amount proceeds in a geomet cal Proportion, wherein the Time ($= t$) or Number of Yea is always to or the ſame with the Index of the Power of th laſt and higheſt Term of the Series, *viz.* R^s or R^t; for i Compound Intereſt the Letters denote as below.

$P =$ the Principal put to Intereſt.
$t =$ the Time of its Continuance.
$A =$ the Amount of the Principal and Intereſt.
$R =$ the Amount of 1*l.* and its Intereſt for one Year,
any given Rate, which may be thus found.

389. Now as one Pound is to the Amount of one Pound fo any given Time,. (t) ſo is any propoſed Principal or Sum to i Amount for the ſame Time.

That is, As $1 : R^t :: P : PR^t$.
But $PR^t = A$, the general Theorem.

From this Theorem any three of the four Quantities P, R, t, A being given, the other may be found. But becauſe t in all the Caſes is the Index, or denotes the Power to which the Quan tity R muſt be raiſed, and is in itſelf uſually large, it plainl appears that in the Solution of this Sort of Queſtions, there wil be great Labour required, if we proceed in the common alge braic Way of extracting of Roots, or raiſing of Powers, which therefore we ſhall here decline, and ſhew how this may be don very eaſily by Logarithms, and alſo by Tables ready computed for ſhewing the Value of R^t for any Number of Days an Years, and at any of the uſual Rates of Intereſt: And firſt by Logarithms.

390. From the Theorem $PR^t = A$, we have $R^t = \frac{A}{P}$

which muſt now be expreſſed logarithmically; thus, $R^t = t \times$ L. R, (here L denotes the Logarithm of the Quantity to which it is prefixed. See 134, 141.) and $\frac{A}{P} =$ L. A — L. P; and fo $t \times$ L. R $=$ L. A — L. P; whence we have the following Theorems.

$t \times L.R + L.P = L.A$, Theorem I.

$L.A - t \times L.R = L.P$, Theorem II.

$\frac{L.A - L.P}{t} = L.R$, Theorem III.

$\frac{L.A - L.P}{L.R} = t$, Theorem IV.

391. We ſhall illuſtrate theſe Theorems by the following Examples.

QUESTION I.

What will 275*l.* 15*s.* amount to in three Years and an half, at $4\frac{1}{2}$ *per Cent. per Ann.* Compound Intereſt?

Here $\begin{cases} P = 275{,}75, \text{ the Principal.} \\ R = 1{,}045, \text{ the Rate of Intereſt.} \\ t = 3{,}5, \text{ the Time.} \\ \text{To find A, the Amount, } \textit{per} \text{ Theor. I.} \end{cases}$

The Logarithm of the Rate . . $R = 1{,}045 = 0{,}019116$
Multiply by the Time . . . . $t = 3{,}5$

The Product is the Logarithm of $R^t = 1{,}1665 = 066906$
To which add the Logarithm of $P = 275{,}75 = 2{,}440515$

The Sum is the Log. of $PR^t = L.A = 321{,}68 = 2{,}507421$

So the Amount ſought is 321*l.* 13*s.* 7*d.* which is more than the Amount by Simple Intereſt by 2*l.* 10*s.* as may be found in Theorem I. Article 270.

QUESTION II.

392. What Principal or Sum being put to Uſe at $4\frac{1}{2}$ *per Cent.* Compound Intereſt, will amount to 321*l.* 13*s.* 7*d.* in three Years and an half?

Here $\begin{cases} P = 321{,}68, \text{ the Principal.} \\ R = 1{,}045 \text{ the Rates of Intereſt.} \\ t = 3{,}5, \text{ the Time.} \\ \text{To find A, the Amount, } \textit{per} \text{ Theor. II.} \end{cases}$

From the Logarithm of . . . $A = 321{,}68 = 2{,}507421$
Subduct the Logarithm of . . $R^t = 1{,}1665 = 0{,}066906$

The Differ. is the Log. of $\frac{A}{R^t} = L.\,P = 275{,}75 = 2{,}440515$

Therefore the Principal ſought is 275*l.* 15*s.*

Question III.

393. At what Rate *per Cent.* &c. Compound Intereſt, will 275*l.* 15*s.* raiſe a Stock, or amount to 321*l.* 13*s.* 7*d.* in three Years and an Half?

$$\text{Here}\left\{\begin{array}{l} P = 321{,}68. \\ R = 275{,}75 \\ t = 3{,}5. \\ \text{To find A, the Amount, } \textit{per} \text{ Theor. III.} \end{array}\right.$$

From the Logarithm of . . . $A = 321{,}68 = 2{,}50742$
Subtract the Logarithm of . . $P = 275{,}75 = 2{,}440515$

The Difference is the Logarithm of R^t . . . $0{,}066906$
Which divide by the Time . . . . . $t = 3{,}5$

The Quotient is the Logarithm of $R = 1{,}045 = 0{,}019116$
Then as $1 : 0{,}045 :: 100 : 4{,}5 = 4\frac{1}{2}l.$ *per Cent.* the Rate required.

Question IV.

394. In what Time will 275*l.* 15*s.* raiſe a Stock of 321*l.* 13*s.* 7*d.* at the Rate of $4\frac{1}{2}l.$ *per Cent.* Compound Intereſt?

$$\text{Here}\left\{\begin{array}{l} P = 275{,}75. \\ R = 321{,}68. \\ t = 1{,}045. \\ \text{To find A, the Amount, } \textit{per} \text{ Theor. IV.} \end{array}\right.$$

From the Logarithm of . . . $A = 321{,}68 = 2{,}507421$
Subduct the Logarithm of . . . $P = 275{,}75 = 2{,}440515$

The Difference is the Logarith. of $R^t = 1{,}045^t = 0{,}066906$

Then the Logar. of 1,045 = 0,019116) 0,066906 (3,5 = t

57348

95580

95580

. . . . .

Thus the Anſwer is three Years and ſix Months.

Note, As the Amount of any Sum and its Intereſt is greater at Compound Intereſt than at Simple Intereſt, for any Time above a Year, ſo it is leſs at Compound than at Simple Intereſt for any Time leſs than a Year, as the Learner may eaſily prove by the Theorems before going.

395. From the above Examples it appears how expeditiouſly Queſtions of Compound Intereſt are ſolved by Logarithms; but as few People concerned in this Affair underſtand this uſeful Method of Computation, we ſhall next exhibit a Set of Tables, wherein the Value of R^t is expreſſed for any Time, or Rate of Intereſt it can be required, and then the whole Buſineſs of Compound Intereſt is done by Multiplication and Diviſion only.

396. The firſt Table expreſſes R^t for Days, and is made from the general Theorem $PR^t = A$; thus, Put $P = 1$*l.* then will the Theorem be $R^t = A$: Suppoſe $R = 1{,}05$*l.* for one Year, or 365 Days, then we have $R^{365} = A =$ Amount for one Day. This 365th Root of R may be extracted algebraically, or be found by a large Table of Logarithms, near enough for Uſe: And if $R = 1{,}05$*l.* then $1{,}05^{365} = 1{,}0003368$*l.* = the Amount of 1*l.* for one Day, at the Rate of 5 *per Cent.* Compound Intereſt.

397. Then ſay: As 1 : 1,00013368 : : 1,00013368 : 1,00026738 = the Amount of 1*l.* for two Days, and ſo on; or thus,

If $R^1 = 1{,}00013368 =$ Amount of 1*l.* for 1 Day.
Multiply by $R = 1{,}00013368$

The Product is $R^2 = 1{,}00026738 =$ Amount for two Days.
Multiply again by $R = 1{,}00013368$

The Product is $R^3 = 1{,}00040110 =$ Amount for three Days.

And thus the Amounts for the Days, at the ſeveral Rates of Intereſt in the firſt Table, are found.

398. In the ſame Manner is the ſecond Table conſtructed for Years; for if $R^t = A = 1{,}05l.$ then ſay as before $1 : 1{,}05 :: 1{,}05 : 1{,}1025 = R^2 =$ Amount of 1*l.* at the End of the ſecond Year; then $1{,}1025 \times 1{,}05 = 1{,}157625 = R^3 =$ Amount of 1*l.* at the End of the third Year; and thus you proceed for the Amounts at the End of every other Year in the Table.

399. Now the Uſe of theſe Tables is extremely eaſy; for let N be put for any Number in the Table; that is, let $R^t = N$, then the Theorem $PR^t = A$, will be $PN = A$; whence we have $P = \frac{A}{N}$; and ſo the *Amount of any Principal is known by multiplying the ſaid Principal by the tabular Number for the given Rate and Time*; and the Amount divided by the ſaid Number gives the Principal.

EXAMPLE.

400. What will 246*l.* amount to in 30 Days, or Years, at 5 *per Centum?*

In Table I. againſt 30 Days, under 5 *per Cent.* ſtands	1,0040182
Which multiplied by the Principal	246
The Product is the Amount required, *viz.*	£. 247,06847
In Table II. againſt 30 Years, and 5 *per Cent.* is	4,3219424
Which multiplied by the Principal	246
Gives the Amount for that Time, *viz.*	£. 1063,1978, &c.

401. By the Reverſe the Principal is found for any given Amount, which needs no Example. If it happens that the Amount be required for any Number of Days or Years that are not in the Tables, then obſerve this

RULE.

RULE.

Divide the given Number of Days, or Years, into two ſuch Numbers as are in the Table, then multiply the Amounts pertaining to each, into each other, the Product will be the Amount for the Time required.

EXAMPLE I.

What will 523*l.* amount to, in 194 Days, at 5 *per Cent. per Annum?*

The two Parts of this Number in the Table are 190 and 4; therefore,

In Table I. againſt 190 Days, under 5 *per Cent.* is	1.0257228
And againſt 4 Days, at the ſame Rate, is - -	1.0005348
The Product is the Amount of 1*l.* for 194 Days, *viz.* - - - - - -	1.0262714
Which multiply by the principal Sum, *viz.* - - -	523
This Product is the Anſwer - - - - *l.*	536.7399840

In Money, 536*l.* 14*s.* 9½*d.*

EXAMPLE II.

What is the Amount of 150*l.* in 91 Years, at 5 *per Cent?*

In Table II. againſt 50 Years, under 5 *per Cent.* is	11.4674000
And againſt 41 Years, at 5 *per Cent.* is - -	7.3919881
The Product is the Amount of 1*l.* for 91 Years, *viz.* - - - - - - - -	84.7668833
Which multiply by the principal Sum, *viz.* - -	150
The Product is the Anſwer - - - *l.*	12715.032495

In Money, 12715*l.* 0*s.* 7½*d.*

EXAMPLE III.

What will 523*l.* amount to in 5 Years and 194 Days, at 5 *per Cent?*

In Table II. againſt 5 Years, at 5 *per Cent.* is	1.2762816
And the Amount of 1*l.* in 194 Days, as above, is	1.0262714
The Product is the Amount of 1*l.* in 5 Years and 194 Days - - - - -	1.3098113
Which multiplied by the principal Sum - -	523
The Product is the Anſwer, *viz.* - - -	*l.* 685.0313413

In Money, 685*l.* 0*s.* 7½*d.*

N. B. The Reaſon of this Rule is, that the Numbers of Days and Years are in arithmetical Progreſſion, and the correſpondent Numbers in the Tables are in geometrical Progreſſion, (396, 397.) and therefore to the *Addition* or *Sum* of any two of the former, there anſwers a *Multiplication*, or a *Product* of the latter. (See 132.)

TABLES of COMPOUND INTEREST.

TABLE I.

The Amount *of one Pound for* Days, *at the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, *and* 6 per Cent. per Annum.

Days.	2 *per Cent.*	2½ *perCent.*	3 *per Cent.*	3½ *perCent*
1	1.0000542	1.0000676	1.0000809	1.0000942
2	1.0001085	1.0001353	1.0001619	1.0001885
3	1.0001627	1.0002029	1.0002429	1.0002827
4	1.0002170	1.0002706	1.0003240	1.0003770
5	1.0002713	1.0003383	1.0004050	1.0004713
6	1.0003255	1.0004059	1.0004860	1.0005656
7	1.0003798	1.0004736	1.0005670	1.0006600
8	1.0004341	1.0005412	1.0006480	1.0007542
9	1.0004884	1.0006090	1.0007291	1.0008486
10	1.0005426	1.0006767	1.0008101	1.0009429
20	1.0010856	1.0013539	1.0016209	1.0018867
30	1.0016289	1.0020315	1.0024324	1.0028315
40	1.0021725	1.0027097	1.0032445	1.0037771
50	1.0027163	1.0033882	1.0040573	1.0047236
60	1.0032605	1.0040673	1.0048708	1.0056710
70	1.0038049	1.0047468	1.0056849	1.0066193
80	1.0043497	1.0054267	1.0064996	1.0075685
90	1.0048947	1.0061071	1.0073151	1.0085186
100	1.0054401	1.0067880	1.0081311	1.0094696
110	1.0059857	1.0074693	1.0089479	1.0104214
120	1.0065316	1.0081511	1.0097653	1.0113742
130	1.0070779	1.0088334	1.0105834	1.0123279
140	1.0076244	1.0095161	1.0114021	1.0132825
150	1.0081712	1.0101993	1.0122215	1.0142379
160	1.0087183	1.0108829	1.0130415	1.0151943

TABLE I.

The Amount of one Pound, Compound-Intereſt.

Days.	4 *per Cent.*	4½ *perCent.*	5 *per Cent.*	6 *per Cent*
1	1.0001074	1.0001206	1.0001336	1.0001596
2	1.0002149	1.0002412	1.0002673	1.0003193
3	1.0003224	1.0003618	1.0004011	1.0004790
4	1.0004299	1.0004824	1.0005348	1.0006387
5	1.0005374	1.0006031	1.0006685	1.0007985
6	1.0006449	1.0007238	1.0008023	1.0009583
7	1.0007524	1.0008445	1.0009361	1.0011181
8	1.0008600	1.0009652	1.0010699	1.0012779
9	1.0009675	1.0010859	1.0012037	1.0014378
10	1.0010751	1.0012066	1.0013376	1.0015976
20	1.0021513	1.0024148	1.0026770	1.0031979
30	1.0032288	1.0036243	1.0040182	1.0048007
40	1.0043074	1.0048354	1.0053611	1.0064060
50	1.0053871	1.0060479	1.0067059	1.0080139
60	1.0064680	1.0072618	1.0080525	1.0096244
70	1.0075501	1.0084773	1.0094009	1.0112375
80	1.0086333	1.0096942	1.0107511	1.0128531
90	1.0097177	1.0109125	1.0121031	1.0144713
100	1.0108033	1.0121324	1.0134569	1.0160921
110	1.0118900	1.0133537	1.0148125	1.0177155
120	1.0129779	1.0145765	1.0161699	1.0193415
130	1.0140670	1.0158007	1.0175291	1.0209701
140	1.0151572	1.0170265	1.0188902	1.0226013
150	1.0162487	1.0182537	1.0202531	1.0242351
160	1.0173412	1.0194824	1.0216178	1.0258715

TABLE I.

The Amount of one Pound, Compound-Interest.

Days.	2 *per Cent.*	2½ *perCent.*	3 *per Cent.*	3½ *perCent.*
170	1.0092658	1.0115670	1.0138623	1.0161516
180	1.0098135	1.0122516	1.0146837	1.0171098
190	1.0103615	1.0129366	1.0155057	1.0180689
200	1.0109098	1.0136221	1.0163284	1.0190288
210	1.0114584	1.0143081	1.0171518	1.0199897
220	1.0120073	1.0149945	1.0179759	1.0209515
230	1.0125565	1.0154814	1.0188006	1.0219142
240	1.0131060	1.0163687	1.0196260	1.0228778
250	1.0136558	1.0170565	1.0204520	1.0238424
260	1.0142059	1.0177448	1.0212788	1.0248078
270	1.0147563	1.0184336	1.0221062	1.0257741
280	1.0153070	1.0191228	1.0229342	1.0267414
290	1.0158580	1.0198125	1.0237630	1.0277096
300	1.0164093	1.0205026	1.0245924	1.0286786
310	1.0169609	1.0211932	1.0254225	1.0296486
320	1.0175127	1.0218843	1.0262532	1.0306195
330	1.0180649	1.0225758	1.0270847	1.0315914
340	1.0186174	1.0232679	1.0279168	1.0325641
350	1.0191702	1.0239603	1.0287495	1.0335378
360	1.0197233	1.0246533	1.0295830	1.0345123
361	1.0197786	1.0247226	1.0296664	1.0346098
362	1.0198340	1.0247919	1,0297497	1.0347073
363	1.0198893	1.0248613	1.0298331	1.0348049
364	1.0199446	1.0249306	1.0299165	1.0349024
365	1.0200000	1.0250000	1.0300000	1.0350000

TABLE I.

The Amount of one Pound, Compound-Intereſt.

Days.	4 *per Cent.*	4½ *perCent.*	5 *per Cent.*	6 *per Cent.*
170	1.0184350	1.0207126	1.0229843	1.0275105
180	1.0195299	1.0219442	1.0243527	1.0291522
190	1.0206261	1.0231774	1.0257228	1.0307964
200	1.0217233	1.0244120	1.0270949	1.0324433
210	1.0228218	1.0256481	1.0284687	1.0340928
220	1.0239215	1.0268858	1.0298444	1.0357450
230	1.0250233	1.0281249	1.0312219	1.0373998
240	1.0261243	1.0293655	1.0326013	1.0390572
250	1.0272275	1.0306076	1.0339825	1.0407174
260	1.0283319	1.0318512	1.0353656	1.0423800
270	1.0294375	1.0330963	1.0367505	1.0440454
280	1.0305443	1.0343429	1.0381373	1.0457135
290	1.0316522	1.0355918	1.0395259	1.0473842
300	1.0327614	1.0368406	1.0409164	1.0490576
310	1.0338717	1.0383917	1.0423087	1.0507336
320	1.0349832	1.0393444	1.0437029	1.0524124
330	1.0360960	1.0405985	1.0450990	1.0540938
340	1.0372099	1.0418542	1.0464969	1.0557779
350	1.0383250	1.0431114	1.0478967	1.0574647
360	1.0394413	1.0443700	1.0492984	1.0591542
361	1.0395530	1.0444960	1.0494387	1.0593233
362	1.0396648	1.0446220	1.0495790	1.0594924
363	1.0397765	1.0447479	1.0497193	1.0596616
364	1.0398882	1.0448739	1.0498596	1.0598308
365	1.0400000	1.0450000	1.0500000	1.0600000

TABLE II.

Of COMPOUND INTEREST.

he Amount *of one Pound for* Years, *at the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, *and* 6, per Cent. per Annum.

Years.	2 *per Cent.*	2½ *perCent.*	3 *per Cent.*	3½ *perCent.*
1	1.0200000	1.0250000	1.0300000	1.0350000
2	1.0404000	1.0506250	1.0609000	1.0712250
3	1.0612080	1.0768906	1.0927270	1.1087178
4	1.0824321	1.1038128	1.1255088	1.1475230
5	1.1040808	1.1314082	1.1592740	1.1876863
6	1.1261624	1.1596934	1.1948523	1.2292553
7	1.1486856	1.1886857	1.2298733	1.2722792
8	1.1716593	1.2184029	1.2667700	1.3168098
9	1.1950925	1.2488629	1.3047731	1.3628973
10	1.2189944	1.2800845	1.3439163	1.4105987
11	1.2433743	1.3120866	1.3842338	1.4599697
12	1.2682417	1.3448888	1.4257608	1.5110686
13	1.2936066	1.3785110	1.4685337	1.5639560
14	1.3194787	1.4129738	1.5125897	1.6186945
15	1.3458683	1.4482981	1.5579674	1.6753488
16	1.3727857	1.4845056	1.6047064	1.7339860
17	1.4002414	1.5216182	1.6528476	1.7946755
18	1.4282462	1.5596587	1.7024330	1.8574892
19	1.4568111	1.5986501	1.7535060	1.9225013
20	1.4859474	1.6386164	1.8061112	1.9897888
21	1.5156663	1.6795818	1.8602945	2.0594314
22	1.5459796	1.7215714	1.9161034	2.1315115
23	1.5768992	1.7646106	1.9735865	2.2061144
24	1.6084372	1.8087259	2.0327941	2.2833284
25	1.6406059	1.8539441	2.0937779	2.3632449

TABLE

TABLE II.

The Amount of one Pound, Compound-Intereſt.

Years	4 *per Cent.*	$4\frac{1}{2}$ *perCent.*	5 *per Cent.*	6 *per Cent.*
1	1.0400000	1.0450000	1.0500000	1.0600000
2	1.0816000	1.0920250	1.1025000	1.1236000
3	1.1248640	1.1411661	1.1576250	1.1910160
4	1.1698586	1.1925186	1.2155063	1.2624769
5	1.2166529	1.2461819	1.2762816	1.3382256
6	1.2653190	1.3022601	1.3400956	1.4185191
7	1.3159318	1.3608618	1.4071064	1.5036303
8	1.3685691	1.4221006	1.4774554	1.5938481
9	1.4233118	1.4860951	1.5513282	1.6894790
10	1.4802443	1.5529694	1.6288946	1.7908477
11	1.5394541	1.6228530	1.7103393	1.8982980
12	1.6010322	1.6958814	1.7958563	2.0121965
13	1.6650735	1.7721961	1.8856491	2.1329283
14	1.7316764	1.8519449	1.9799316	2.2609039
15	1.8009435	1.9352824	2.0789282	2.3965582
16	1.8729812	2.0223701	2.1828746	2.5402517
17	1.9479005	2.1133768	2.2920183	2.6927728
18	2.0258165	2.2084787	2.4066192	2.8543392
19	2.1068492	2.3078603	2.5269502	3.0255995
20	2.1911231	2.4117140	2.6532977	3.2071355
21	2.2787681	2.5202111	2.7859626	3.3995636
22	2.3699188	2.6336520	2.9252607	3.6035374
23	2.4647155	2.7521663	3.0715238	3.8197497
24	2.5633042	2.8760838	3.2251000	4.0489346
25	2.6658363	3.0054344	3.3863549	4.2918707

TABLE II.

The Amount of one Pound, Compound-Intereſt.

Years.	2 *per Cent.*	2½ *perCent.*	3 *per Cent.*	3½*perCent.*
26	1.6734181	1.9002927	2.1565912	2.4459585
27	1.7068864	1.9478000	2.2212890	2.5315871
28	1.7410242	1.9964950	2.2879276	2.6201719
29	1.7758446	2.0464073	2.3565655	2.7118779
30	1.8113615	2.0975675	2.4272624	2.8067937
31	1.8475888	2.1500067	2.5000803	2.9050314
32	1.8845405	2.2037569	2.5750827	3.0067075
33	1.9222314	2.2588508	2.6523352	3.1119423
34	1.9606760	2.3153221	2.7319053	3.2208603
35	1.9998895	2.3732051	2.8138624	3.3335904
36	2.0398873	2.4325353	2.8982783	3.4502661
37	2.0806850	2.4933487	2.9852266	3.5710254
38	2.1222987	2.5556824	3.0747834	3.6960113
39	2.1647447	2.6195744	3.1670269	3.8253717
40	2.2080396	2.6850638	3.2620377	3.9592597
41	2.2522004	2.7521904	3.3598989	4.0978338
42	2.2979444	2.8209952	3.4606958	4.2412579
43	2.3431893	2.8915500	3.5645167	4.3897020
44	2.3900531	2.9638080	3.6714522	4.5433416
45	2.4378542	3.0379032	3.7815958	4.7023585
46	2.4866112	3.1138508	3.8950437	4.8669411
47	2.5363435	3.1916971	4.0118950	5.0372840
48	2.5870703	3.2714895	4.1322518	5.2135889
49	2.6388117	3.3532768	4.2562194	5.3960645
50	2.6915880	3.4371087	4.3839060	5.5849268

TABLE II.

The Amount of one Pound, Compound-Intereſt.

Years.	4 *per Cent.*	4½ *perCent.*	5 *per Cent.*	6 *per Cent.*
26	2.7724697	3.1406790	3.5556727	4.5493829
27	2.8833685	3.2820095	3.7334563	4.8223459
28	2.9987033	3.4296999	3.9201291	5.1116866
29	3.1186514	3.5840364	4.1161356	5.4183878
30	3.2433975	3.7453181	4.3219424	5.7434911
31	3.3731334	3.9138574	4.5380395	6.0881006
32	3.5080587	4.0899810	4.7649415	6.4533866
33	3.6483811	4.2740301	5.0031885	6.8405898
34	3.7943163	4.4663615	5.2533480	7.2510252
35	3.9460889	4.6673478	5.5160154	7.6860869
36	4.1039325	4.8773784	5.7918161	8.1472519
37	4.2680898	5.0968604	6.0814069	8.6360870
38	4.4388134	5.3262192	6.3854773	9.1542523
39	4.6163659	5.5658990	6.7047511	9.7035074
40	4.8010206	5.8163645	7.0399887	10.2857178
41	4.9930614	6.0781009	7.3919881	10.9028609
42	5.1927839	6.3516154	7.7615875	11.5570326
43	5.4004952	6.6374381	8.1496669	12.2504545
44	5.6165150	6.9361229	8.5571503	12.9854818
45	5.8411756	7.2482484	8.9850078	13.7646107
46	6.0748227	7.5744196	9.4342582	14.5904873
47	6.3178156	7.9152684	9.9059711	15.4659166
48	6.5705282	8.2714555	10.4012696	16.3938716
49	6.8333493	8.6436710	10.9213331	17.3775039
50	7.1066833	9.0326362	11.4674000	18.4201541

402. The Conſtruction of Table III. is by Theorem II. *viz.* $\frac{A}{R^t} = P$, the preſent Worth or Value of one Pound, which is ere to be conſidered as the Amount; therefore if A = 1*l.* R = 1,05, and $t = 1, 2, 3, 4, 5$, and Years, as before; 'tis eident that Unity, or 1, being divided by the Numbers in the econd Table (deſigned by R^t) will give the Numbers in this hird Table, or the preſent Values of 1*l.* for the *Tabular Years*, nd 5 *per Cent.* and ſo for any other Rate of Intereſt.

Example at 5 per Cent.

Unity or 1 divided by	1,05	The Quot. is	,952381	The preſent Value of 1*l.* for	1	Years, &c.
	1,1025		,907029		2	
	1,157625		,863837		3	
	1,21550625		,822702		4	
	1,276281		,783526		5	

403. The Uſe of the Table is very eaſy, as will appear from he following Example.

What is the preſent Worth of 246*l.* due at the End of 30 ears, allowing 5 *per Cent.* Compound Intereſt?

n Table III. againſt the given Time and Rate, is the Number	0,2313775
hich multiply by the given Sum - -	246
he Product is the *preſent Worth* required	£. 56,9189, &c.

404. Note, in this Table you have the Numbers for whole ears only; but if the Time conſiſts of Years, and Parts of a ear, you take the Difference between the two contiguous hole Years, and then a proportional Part of that Difference ubducted from the tabular Number of the given Years, will e the tabular Number required for the given Time.

TABLE III.

Of COMPOUND INTEREST,

405. *The* preſent Worth *of one Pound for* Years, *the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, *and* per Cent. per Annum.

Years.	*2 per C.*	*2½ per C.*	*3 per C.*	*3½ perC.*
1	.9803921	.9756097	.9708738	.9661836
2	.9611687	.9518144	.9425959	.9335107
3	.9423223	.9285994	.9151417	.9019427
4	.9238454	.9059506	.8884870	.8714422
5	.9057308	.8838542	.8626088	.8419732
6	.8879713	.8622968	.8374843	.8135006
7	.8705601	.8412653	.8130915	.7859910
8	.8534903	.8207465	.7894092	.7594116
9	.8367552	.8007283	.7664167	.7337719
10	.8203483	.7811984	.7440939	.7089188
11	.8042630	.7621447	.7224213	.6849457
12	.7884931	.7435558	.7013799	.6617833
13	.7730325	.7254203	.6809513	.6394041
14	.7578750	.7077272	.6611178	.6177818
15	.7430147	.6904655	.6418619	.5968906
16	.7284458	.6736249	.6231669	.5767059
17	.7141625	.6571950	.6050164	.5572038
18	.7001593	.6411659	.5873946	.5383611
19	.6864307	.6255277	.5702860	.5201557
20	.6729713	.6102709	.5536758	.5025659
21	.6597758	.5953862	.5375493	.4855709
22	.6468390	.5808646	.5218925	.4691506
23	.6341559	.5666972	.5066917	.4532856
24	.6217214	.5528753	.4919337	.4379571
25	.6095308	.5393905	.4776056	.4231470

TABL

TABLE III.

The preſent Worth of one Pound, Compound-Intereſt.

Years.	4 *per C.*	4½ *per C.*	5 *per C.*	6 *per C.*
1	.9615385	.9569378	.9523810	.9433962
2	.9245562	.9157299	.9070295	.8899964
3	.8889964	.8762966	.8638376	.8396193
4	.8548042	.8385613	.8227025	.7920937
5	.8219271	.8024511	.7835262	.7472582
6	.7903145	.7618957	.7462154	.7049605
7	.7599178	.7348285	.7106813	.6650571
8	.7306902	.7031851	.6768394	.6274124
9	.7025867	.6729044	.6446089	.5918985
10	.6755642	.6439277	.6139133	.5583948
11	.6495809	.6161987	.5846793	.5267875
12	.6245971	.5896639	.5568374	.4969694
13	.6005741	.5642716	.5303214	.4688390
14	.5774751	.5399729	.5050679	.4423010
15	.5552645	.5167204	.4810171	.4172651
16	.5339082	.4944693	.4581115	.3936463
17	.5133733	.4731764	.4362967	.3713644
18	.4936281	.4528004	.4155207	.3503438
19	.4746424	.4333018	.3957340	.3305130
20	.4563870	.4146429	.3768895	.3118047
21	.4388336	.3967874	.3589424	.2941554
22	.4219554	.3797009	.3418499	.2775051
23	.4057263	.3633501	.3255713	.2617973
24	.3901215	.3477035	.3100679	.2469786
25	.3751168	.3327306	.2953028	.2329986

TABLE

TABLE III.

The preſent Worth of one Pound, Compound-Intereſt.

Years.	2. *per C.*	2½ *per C.*	3. *per C.*	3½ *per C.*
26	.5975793	.5262347	.4636947	.4088378
27	.5858620	.5133997	.4501891	.3950123
28	.5743746	.5008778	.4370768	.3816543
29	.5631123	.4886613	.4243464	.3687482
30	.5520709	.4767427	.4119868	.3562784
31	.5412460	.4651148	.3999871	.3442304
32	.5306333	.4537706	.3883379	3325897
33	.5202287	.4427030	.3770263	.3213427
34	.5100282	.4319053	.3660449	.3104761
35	.5000276	.4213711	.3553834	.2999769
36	.4902232	.4110937	.3450324	.2898327
37	.4806109	.4010671	.3349829	.2800316
38	.4711872	.3912849	.3252262	.2705619
39	.4619482	.3817414	.3157536	.2614125
40	.4528904	.3724306	.3065568	.2525725
41	.4440102	.3633470	.2976280	.2440314
42	.4353041	.3544848	.2889592	.2357791
43	.4267688	.3458389	.2805429	.2278059
44	.4184008	.3374038	.2723718	.2201023
45	.4101968	.3291744	.2644386	.2126594
46	.4021537	.3211458	.2567395	.2054679
47	.3942684	.3133129	.2492588	.1985197
48	.3865376	.3056712	.2419988	.1918065
49	.3789584	.2982158	.2349503	.1853202
50	.3715279	.2909422	.2281071	.1790534

TABLE III.

The preſent Worth of one Pound, Compound-Intereſt.

Years.	4 per C.	4½ per C.	5 per C.	6 per C.
26	.3606892	.3184025	.2812407	.2198100
27	.3468166	.3046914	.2678483	.2073680
28	.3334775	.2915707	.2550936	.1956301
29	.3206514	.2790150	.2429463	.1845567
30	.3083187	.2670000	.2313775	.1741101
31	.2964603	.2555024	.2203595	.1642548
32	.2850579	.2444999	.2098662	.1549574
33	.2740942	.2339712	.1998762	.1461862
34	.2635521	.2238959	.1903548	.1379115
35	.2534155	.2142544	.1812903	.1301052
36	.2436687	.2050282	.1726574	.1227408
37	.2342969	.1961992	.1644356	.1157932
38	.2252854	.1877504	.1566054	.1092389
39	.2166206	.1796655	.1491479	.1030555
40	.2082890	.1719287	.1420457	.0972222
41	.2002779	.1645251	.1352816	.0917191
42	.1925749	.1574403	.1288396	.0865274
43	.1851682	.1506605	.1227044	.0816296
44	.1180464	.1441728	.1168613	.0770091
45	.1711984	.1379644	.1112965	.0726501
46	.1646139	.1320233	.1059967	.0685378
47	.1582826	.1263381	.1009492	.0646583
48	.1521948	.1208977	.0961421	.0609984
49	.1463411	.1156916	.0915639	.0575457
50	.1407126	.1107097	.0872037	.0542884

406. Theorems *reſolving all Queſtions relating to* Annuities, *&c.* *in* Arrear, *calculated at* Compound Intereſt.

Here $\begin{cases} R = \text{one Pound, and its Intereſt for one Year, as before.} \\ U = \text{the firſt Year's Rent without Intereſt.} \end{cases}$

Then RU = the Amount of the firſt Year's Rent, and its Intereſt.

And hence is form'd the following Progreſſion of Amounts in continued geometrical Proportion.

$$\text{Thus}\begin{cases} \;1. \quad\;\; 2. \quad\;\;\; 3. \quad\;\;\; 4. \quad\;\;\; 5. \quad \&c. \text{ the Years.} \\ U + UR + UR^2 + UR^3 + UR^4, \; \&c. \text{ the Amounts.} \end{cases}$$

Hence $U + UR + UR^2 + UR^3 + UR = A$, the Amount of any yearly Rent, or Annuity, forborn five Years.

Now the laſt Term in the above Series is $UR^4 = UR^{t-1}$.

Therefore $A - UR^{t-1}$ = the Sum of all the Antecedents.

And $A - U$ = the Sum of all the Conſequents in the Series.

So that it will be, $U : RU :: A - UR^{t-1} : A - U$.*

Therefore $AU - UU = RUA - UUR^t$. Divide all by U.

Then $A - U = RA - UR^t$. The General Theorem.

407. From hence we deduce the following particular Theorems, *viz.*

Given U, R, *t*, to find A?

$$\text{Theorem 1.} \quad \frac{UR^t - U}{R - 1} = A.$$

Given A, R, *t*, to find U?

$$\text{Theorem 2.} \quad \frac{RA - A}{R^t - 1} = U.$$

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* For in any geometrical Series, it will be as any one Antecedent is to its Conſequent, ſo is the Sum of all the Antecedents to the Sum of all the Conſequents; thus, let the Series be $a : ar :: ar : ar^2 :: ar^2 : ar^3 :: ar^3 : ar^4$, *&c.* Then as $a : ar :: a + ar + ar^2 + ar^3$, *&c.* $: ar + ar^2 + ar^3 + ar^4$, &c. for the Product of the two Extremes is evidently equal to the Product of the two Means. (138)

Given U, A, R, to find t?

Theorem III. $\frac{RA + U - A}{U} = R^t$.

Given A, U, t, to find R?

Theorem IV. $\frac{A}{U} R - R^t = \frac{A - U}{U}$

QUESTION I.

408. If 30l. yearly Rent be forborn, or unpaid nine Years, what will it amount to at the Rate of 6*l.* *per Cent.* &c. Compound Intereſt?

Here is given $\left\{\begin{array}{l} U = 30 \\ t = 9 \\ R = 1{,}06 \end{array}\right\}$ To find A, *per* Theorem I.

In the firſt Place, let R = - - $1{,}06 = 0{,}025305$
Be involved to the 9th Power (*viz.* R^t) - - - 9
That will be - - $R^9 = 1{,}689451 = 0{,}227745$
Multiply by - - - $U = 30 = 1{,}477121$
The Product is - - $U R^t = 50{,}683530 = 1{,}704866$
From that ſubtract - - $U = 30$
The Remainder is the Dividend $= 20{,}68353 = UR^t - U$.
Divide therefore - $U R^t - U = 20{,}68353 = 1{,}315626$
By - - - - $R - 1 = 0{,}06 = 8{,}778151$
The Quotient is - - $A = l.\ 344{,}7267 = 2{,}537475$

That is the Amount = 344*l.* 14*s.* 6¼*d.* the Anſwer.

QUESTION II.

409. What Annuity 3*l.* 10*s.* *per Cent.* Compound-Intereſt, will raiſe a Stock of 344*l.* 5*s.* being forborn eight Years?

Here is given $\left\{\begin{array}{l} A = 344{,}25 \\ R = 1{,}035 \\ t = 8 \end{array}\right\}$ To find U, *per* Theorem II.

Multiply the Amount - - $A = 344,25 = 2,53687$
By the Rate - - - - $R = 1,035 = 0,01494$

From that Product - $RA = 356,29875 = 2,55181$
Subtract the Amount - $A = 344,25$

The Remainder is - $RA - A = 12,04875$, the Dividend

Then involve - - - $R = 1,035 = 0,01494$
To the 8th Power, *viz.* R^t - - - -

That Power will be - - $R^8 = 1,316803 = 0,11952$
The ſame leſs Unity is $R^t - 1 = 0,316803$, the Diviſor

Therefore divide - $RA - A = 12,04875 = 1,08090$
By - - - $R^t - 1 = 0,316803 = 9,50078$

The Quotient is - - $U = 38,0297 = 1,580123$

The Annuity therefore which was ſought, is found to be 38,0297*l.* = 38*l.* 0*s.* 7*d.* *per Ann.* Anſwer.

QUESTION III.

410. In what Time will 38*l.* 0*s.* 7*d.* raiſe a Stock of 344*l.* 5*s.* at 3*l.* 10*s.* *per Cent.* *per Ann.* Compound-Intereſt?

$$\text{Here is given}\left\{\begin{array}{l} U = 38,0297 \\ A = 344,25 \\ R = 1,035 \end{array}\right\}\text{To find } t, \textit{ per } \text{Theorem III.}$$

Firſt multiply the Amount - $A = 344,25 = 2,536874$
By the given Rate - - $R = 1,035 = 0,014940$

To that Product - $RA = 356,29875 = 2,551814$
Add the Annuity - - $U = 38,0297$

From the Sum - $RA + U = 394,32845$
Take the Amount - $A = 344,25$

The Remainder is the Dividend } $RA + U - A = 50,07845 = 1,699651$
Which divide by - - $U = 38,0297 = 1,580123$
The Quotient will be - $R^t = 1,316803 = 0,119528$

Then divide 1,316803 continually by the Rate 1,035 until othing remains, and the Number of thoſe Diviſions will be 8 $= t =$ Time required.

But much better by the Numbers in Table II. where under he given Rate 1,035, you will find the Number 1,316809, gainſt which, by the Side, is eight Years, the Time required. 396, 397, 398.)

QUESTION IV.

411. At what Rate *per Cent.* Compound-Intereſt, will 30*l.* early Rent, being forborn, or unpaid, nine Years, amount to 44*l.* 14*s.* 6¼*d.*

$$\text{Here is given}\left\{\begin{array}{l} U = 30 \\ A = 344{,}7267 \\ t = 9 \end{array}\right\}\text{To find R, } per \text{ Theor. IV.}$$

irſt divide the Amount - $A = 344{,}7267 = 2{,}537475$

y the Annuity - - $U = 30 = 1{,}477121$

he Quotient is - - $\frac{A}{U} = 11{,}4909 = 1{,}000354$

gain the Amount leſs the Annuity is - - $\Big\}\ A - U = 314{,}7267 = 2{,}497933$

hich divide by - - $U = 30 = 1{,}477121$

he Quotient is - $\frac{A - U}{U} = 10{,}49014 = 1{,}020812$

Now $R^t = R^9$. Therefore the Theorem affords this Equa-on, *viz.* $11{,}4909\,R - R^9 = 10{,}49014$.

This being a very high Equation, requires the Aſſiſtance of lgebra, to be found farther on; but as this Value of R may nearly determined by the Tables, with the utmoſt Eaſe, I all next proceed to ſhew

The Conſtruction of TABLE IV.

412. The Conſtruction of Table IV. is from Theorem I. of Annuities, *&c.* in Arrears, *viz.* $\frac{UR^t - U}{R - 1} = A$. Now as it is $U = 1$*l.* and $R = 1,05$, as before, then the Theorem will be brought to $\frac{R^t - 1}{,05} = A$, the Amount of 1*l.* Annuity for the Number of Years deſigned by *t*. That is, from (R^t) the Numbers in the ſecond Table, ſubtract Unity, or 1. The Remainder, divided by ,05 (or R — 1) gives the Numbers in the fourth Table.

Example at 5 *per Cent.*

The Numbers in the 1ſt Table leſs Unity or 1, are	Which divided by ,05, the Quotients are	Amount of 1*l* Annuity for Years, &c.
,05	1,00000	1
,1025	2,05	2
,157625	3,1525	3
,21550625	4,310125	4
,27628156	5,525631	5

And thus you proceed for any other Rate of Intereſt.

413. The Uſe of this Table take in the following Inſtances.

Againſt 9 Years, and under 6 *per Cent.* you find the Number - - - - 11,4913162

Which multiply by the Annuity - - - - 30

The Product is the preſent Worth - £. 344,7394860

Which is nearly the ſame as *per* Theorem I, (408.)

414. Again; in the Equation $11,4909\ R - R^9 = 10,49014$ (411.) it will be eaſy to ſee, that R muſt be ſuch a Rate in Table II. againſt 9 Years, that the Number anſwering to it, added to 10,49014, muſt be but little leſs than 11,4909; becauſe $11,4909\ R = 10,49014 + R^9$: If we but ſlightly examine

xamine the Numbers under 4, 4½, and 5 *per Cent.* we find hey are too ſmall; if we take that under 6 *per Cent.* *viz.* R^9 = 1,689479, and add to it 10,49014, the Sum is 12,17962, nd 11,4909 × 1,06 = 12,1803, *&c.* nearly the ſame; and herefore ſhews the Rate allowed was ſix *per Cent.* as required.

This Inſtance is ſufficient to ſhew the extreme Uſe of Tables, not only to thoſe who do not underſtand Algebra, but ven to thoſe who do.

TABLE IV.

Of COMPOUND INTEREST.

415. *The* Amount *of one Pound* per Ann. *or* Annuity, *for* Years, *at the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, *and* 6, per Cent. per Annum.

Years.	2 *per Cent.*	2½ *perCent.*	3 *per Cent.*	3½ *perCent.*
1	1.0000000	1.0000000	1.0000000	1.0000000
2	2.0200000	2.0250000	2.0300000	2.0350000
3	3.0604000	3.0756230	3.0909000	3.1062250
4	4.1216080	4.1525156	4.1836270	4.2149429
5	5.2040402	5.2563285	5.3091358	5.3624659
6	6.3081210	6.3877367	6.4684099	6.5501522
7	7.4342834	7.5474302	7.6624622	7.7794075
8	8.5829691	8.7361159	8.8923360	9.0516866
9	9.7546284	9.9545188	10.1591061	10.3684958
10	10.9497210	11.2033818	11.4638793	11.7313931
11	12.1687154	12.4834663	12.8077957	13.1419919
12	13.4120897	13.7955530	14.1920296	14.6019616
13	14.6803315	15.1404418	15.6177904	16.1130303
14	15.9739381	16.5189528	17.0863242	17.6769864
15	17.2934169	17.9319267	18.5989139	19.2956809
16	18.6392853	19.3802248	20.1568813	20.9710297
17	20.0120719	20.8647304	21.7615877	22.7050158
18	21.4123124	22.3863487	23.4144354	24.4996913
19	22.8405586	23.9460074	25.1168684	26.3571805
20	24.2973698	25.5446576	26.8703745	28.2796818
21	25.7833172	27.1832740	28.6764857	30.2694707
22	27.2989835	28.8628559	30.5367803	32.3289022
23	28.8440632	30.5844273	32.4528837	34.4604137
24	30.4218625	32.3490379	34.4264702	36.6665282
25	32.0302997	34.1577639	36.4592643	38.9498567

TABLE

TABLE IV.

The Amount of one Pound, Annuity, Compound-Interest.

Years.	4 *per C.*	4½ *per C.*	5 *per C.*	6 *per C.*
1	1.0000000	1.0000000	1.0000000	1.0000000
2	2.0400000	2.0450000	2.0500000	2.0600000
3	3.1216000	3.1370250	3.1525000	3.1836000
4	4.2464640	4.2781911	4.3101250	4.3746016
5	5.4163226	5.4707097	5.5256312	5.6370930
6	6.6329755	6.7168917	6.8019128	6.9753187
7	7.8982945	8.0191518	8.1420084	8.3938378
8	9.2142263	9.3800136	9.5491089	9.8974681
9	10.5827953	10.8021142	11.0265643	11.4913162
10	12.0061071	12.2882094	12.5778925	13.1807958
11	13.4863514	13.8411788	14.2067871	14.9716435
12	15.0258055	15.4640318	15.9171265	16.8699420
13	16.6268377	17.1599133	17.7129829	18.8821385
14	18.2919112	18.9321094	19.5986320	21.0150667
15	20.0235876	20.7840543	21.5785636	23.2759707
16	21.8245311	22.7193367	23.6574918	25.6725289
17	23.6975124	24.7417069	25.8403664	28.2128806
18	25.6454129	26.8550837	28.1323847	30.9056534
19	27.6712294	29.0635625	30.5390039	33.7599925
20	29.7780786	31.3714228	33.0659541	36.7855920
21	31.9692017	33.7831368	35.7192518	39.9927275
22	34.2479698	36.3033779	38.5052144	43.3922911
23	36.6178886	38.9370299	41.4304751	46.9958285
24	39.0826041	41.6891963	44.5019989	50.8155782
25	41.6459083	44.5652101	47.7270988	54.8645128

TABLE IV.

The Amount of one Pound, Annuity, Compound Interest.

Years.	2 *per C.*	2½ *per C.*	3 *per C.*	3½ *per C.*
26	33.6709057	36.0117080	38.5530422	41.3131017
27	35.3443238	37.9120007	40.7096335	43.7590602
28	37.0512103	39.8598008	42.9309225	46.2906273
29	38.7922345	41.8562958	45.2188502	48.9107993
30	40.5680792	43.9027032	47.5754157	51.6226773
31	42.3794408	46.0002707	50.0026782	54.4294719
32	44.2270296	48.1502775	52.5027585	57.3345025
33	46.1115702	50.3540345	55.0778413	60.3412101
34	48.0338016	52.6128653	57.7301765	63.4531524
35	49.9944776	54.9282074	60.4620818	66.6740127
36	51.9943672	57.3014126	63.2759443	70.0076032
37	54.0342545	59.7339479	66.1742226	73.4578693
38	56.1149396	62.2272966	69.1594493	77.0288947
39	58.2372384	64.7829791	72.2342327	80.7249060
40	60.4019832	67.4025535	75.4012597	84.5502778
41	62.6100228	70.0876174	78.6632975	88.5095375
42	64.6822233	72.8398078	82.0231964	92.6073713
43	67.1594678	75.6608030	85.4838923	96.8486293
44	69.5026511	78.5523231	89.0484191	101.2383313
45	71.8927103	81.5161312	92.7198614	105.7816729
46	74.3305645	84.5540344	96.5014172	110.4840315
47	76.8171758	87.6678853	100.3965009	115.3549720
48	79.3535193	90.8595824	104.4083960	120.3882560
49	81.9405697	94.1310729	108.5406479	125.6018450
50	84.5794015	97.4843488	112.7968673	130.9979100

TABLE IV.

The Amount of one Pound, Annuity, Compound-Interest.

Years.	4 *per Cent.*	4½ *perCent.*	5 *per Cent.*	6 *per Cent.*
26	44.3117446	47.5706446	51.1134538	59.1563827
27	47.0842144	50.7113236	54.6691265	63.7057657
28	49.9675830	53.9933332	58.4025828	68.5281116
29	52.9662863	57.4230332	62.3227119	73.6397983
30	56.0849377	61.0070698	66.4388475	79.0581862
31	59.3283352	64.7523878	70.7607899	84.8016774
32	62.7014687	68.6662452	75.2988294	90.8897780
33	66.2095274	72.7562263	80.0637708	97.3431647
34	69.8579045	77.0302565	85.0669594	104.1837546
35	73.6522248	81.4966180	90.3203073	111.4347799
36	77.5983138	86.1639658	95.8363227	119.1208667
37	81.7022464	91.0413443	101.6281388	127.2681187
38	85.9703362	96.1382048	107.7095458	135.9042058
39	90.4091497	101.4644249	114.0950231	145.0584581
40	95.0255157	107.0303231	120.7997742	154.7619656
41	99.8265363	112.8466876	127.8397829	165.0476836
42	104.8195978	118.9247885	135.2317511	175.9505446
43	110.0123817	125.2764040	142.9933386	187.5075772
44	115.4128169	131.9138422	151.1430056	199.7580319
45	121.0293920	138.8499651	159.7001559	212.7435138
46	126.8705677	146.0982135	168.6851637	226.5081246
47	132.9453904	153.6726331	178.1194218	241.0986121
48	139.2632060	161.5879016	188.0253929	256.5645288
49	145.8337342	169.8593572	198.4266626	272.9584006
50	152.6670836	178.5030282	209.3479957	250.3359046

416. Theorems reſolving all Queſtions relating to the *preſe* *Worth* of *Annuities*, *Penſions*, or *Leaſes*, to continue any Time, or in *Reverſion* after a given Time.

The Buſineſs will be to get a Theorem which may conta P, the *preſent Worth*; for this Purpoſe, we muſt have Recourſe to thoſe Theorems which give the Value of A, the Amount, as formerly in *Simple Intereſt*. Now, in Article 389, we had $P R^t = A$; and in Article 407, we had $\frac{U R^t - U}{R - 1} = A$; therefore by equating theſe two Theorems, we get this general Theorem, *viz.*

$$P R^t = \frac{U R^t - U}{R - 1}$$

417. From whence we derive the following particular Theorems.

Given U, R, *t*, to find P?

$$\text{Theorem I.}\left\{\frac{U - \frac{U}{R^t}}{R - 1} = P.\right.$$

Given P, R, *t*, to find U?

$$\text{Theorem II.}\left\{\frac{P R^t \times R - P R^t}{R^t - 1} = U.\right.$$

Given U, R, P, to find *t*?

$$\text{Theorem III.}\left\{\frac{U}{P + U - P R} = R^t.\right.$$

Given U, P, *t*, to find R?

$$\text{Theorem IV.}\left\{\frac{U}{P} = \frac{U}{P} R^t + R^t - R^t + 1.\right.$$

QUESTION I.

418. What is 30*l.* yearly Rent, worth in ready Money, for ts Continuance 7 Years, allowing 6*l.* *per Cent.* Compound Inereſt, to the Purchaſer ?

$$\text{Here is given}\left\{\begin{array}{l}U = 30\\ R = 1,06\\ t = \;\;7\end{array}\right\}\text{to find P, } \textit{per} \text{ Theorem I.}$$

irſt, involve - - - - $R = 1,06 = 0,025305$

o the 7th Power, (*viz.* R^7) - - - - 7

hat will be - - - $R^7 = 1,50061 = 0,177135$

hen divide - - - - - $U = 30 = 1,477121$

y R^t, there will remain - $\frac{U}{R^t} = 19,9520 = 1,299986$

hen from the Annuity - $U = 30$

ubtract - - - - - $\frac{U}{R_t} = 19,952$

emains the Dividend $U - \frac{U}{R^t} = 10,048 = 1,002079$

hich divide by the Rate leſs Unity = R — 1 } $= 0,06 = 8,778151$

he Quotient is the preſent Worth — $P = 167,4716 = 2,223928$

The preſent Worth, in ready Money, is 167*l.* 9*s.* 5*d.* the ſwer.

419. Suppoſe this were an Annuity in Reverſion, or not to entered on till after ſeven Years are paſt, and thence to conue ſeven Years; and you would know the preſent Worth; d by the ſecond Theorem of Compound Intereſt (392.) what dy Money will amount to 167*l.* 9*s.* 5*d.* in ſeven Years, at ſame Rate of Intereſt; and that will be its preſent Worth; ſo for any other Annuity in Reverſion.

The Conſtruction of TABLE V.

420. It is but too eaſy to be obſerved, how very difficult th Solution of theſe very uſeful Queſtions muſt be in this commo Way of Computation, and therefore we ſhall next ſhew, ho a fifth Table may be conſtructed to facilitate this Affair. Thi is done from the general Theorem (416.)

$$P R^t = \frac{U R^t - U}{R - 1}$$

Whence we have $P = \frac{U R^t - U}{R^t \times R - 1} = \frac{R^t - 1}{0{,}05\, R^t}$ (by puttin $U = 1l.$ and $R = 1{,}05$.) But $\frac{R^t - 1}{0{,}05}$ make the Number of Table IV. (412.) and R^t gives the Numbers of Table I (396.) therefore the Numbers of Table V. ariſe from thoſe Table IV. divided by the Numbers in Table II. as in the fol lowing Examples at the Rate of 1,05 *per Cent.*

Table II.	Table IV.	Table V.
1,05)	1,00000	(= 0,95238, &c. for the 1ſt Year.
1,1025)	2,05000	(= 1,85941, &c. for the 2d Year.
1,157625)	3,15250	(= 2,72324, &c. for the 3d Year.
1,21550625)	4,310125	(= 3,54595, &c. for the 4th Year.
1,27628156)	5,525631	(= 4,32947, &c. for the 5th Year, &

And thus for any other Rate of Intereſt.

The Uſe of TABLE V.

421. What is the preſent Worth of 30*l.* *per Annum*, to continue ſeven Years, at 6*l.* *per Cent?*

In the Table againſt 7 Years, and 6*l.* *per Cent.* you find - - - - - - -	5,5823815
Which multiply by the Annuity 30*l.* - - -	30
The Product is the preſent Worth -	= 167,471445

Or 167*l.* 9*s.* 5*d.* as before. (418.)

This one Example is ſufficient to ſhew the Uſe of this Table in any other Caſe.

TABLE V.

Of COMPOUND INTEREST.

422. *The* preſent Worth *of* 1l. per Ann. *or* Annuity, *for* Years, *at the Rates of* 2, 2½, 3, 3½, 4, 4½, 5, *and* 6, per Cent. per Annum.

Years.	2 *per Cent.*	2½ *perCent.*	3 *per Cent.*	3½ *perCent.*
1	0.9803922	0.9756098	0.9708738	0.9661836
2	1.9415609	1.9274242	1.9134697	1.8996943
3	2.8838833	2.8560236	2.8286114	2.8016379
4	3.8077287	3.7619742	3.7170984	3.6730792
5	4.7134595	4.6458285	4.5797072	4.5150524
6	5.6014309	5.5081254	5.4171914	5.3285530
7	6.4719911	6.3493906	6.2302829	6.1145439
8	7.3254814	7.1701372	7.0196922	6.8739555
9	8.1622367	7.9708655	7.7861089	7.6076865
10	8.9825850	8.7520639	8.5302028	8.3166053
11	9.7868480	9.5142087	9.5256241	9.0015510
12	10.5753412	10.2577646	9.9540040	9.6633343
13	11.3483737	10.9831839	10.6349553	10.3027385
14	12.1062487	11.6909122	11.2960731	10.9205203
15	12.8492635	12.3813777	11.9379351	11.5174109
16	13.5777093	13.0550027	12.5611020	12.0941168
17	14.2918719	13.7121977	13.1661185	12.6513206
18	14.9920313	14.3533636	13.7535131	13.1896812
19	15.6784620	14.9788913	14.3237991	13.7098374
20	16.3514333	15.5891623	14.8774748	14.2124033
21	17.0112092	16.1845486	15.4150241	14.6979742
22	17.6580482	16.7654132	15.9369166	15.1671248
23	18.2922041	17.3321105	16.4436084	15.6204105
24	18.9139256	17.8849858	16.9355421	16.0583676
25	19.5234565	18.4243764	17.4131477	16.4815146

TABLE

TABLE V.

The preſent Worth of one Pound, Annuity, Compound-Intereſt.

Years.	4 *per C.*	4½ *per C.*	5 *per C.*	6 *per C.*
1	0.9615385	0.9569378	0.9523809	0.9433962
2	1.8860947	1.8726678	1.8594103	1.8333926
3	2.7750910	2.7489644	2.7232480	2.6730119
4	3.6298952	3.5875257	3.5459505	3.4651056
5	4.4518223	4.3899767	4.3294767	4.2123638
6	5.2421369	5.1578725	5.0756921	4.9173244
7	6.6020547	5.8927009	5.7863734	5.5823815
8	6.7327448	6.5958861	6.4632128	6.2097939
9	7.4353314	7.2687905	7.1078217	6.8016923
10	8.1108955	7.9127182	7.7217349	7.3600871
11	8.7604763	8.5289169	8.3064142	7.8868747
12	9.3850733	9.1185808	8.8632516	8.3838440
13	9.9856473	9.6828524	9.3925730	8.8526831
14	10.5631223	10.2228253	9.8986409	9.2949840
15	11.1183868	10.7395457	10.3796500	9.7122491
16	11.6522949	11.2340151	10.8377695	10.1058953
17	12.1656680	11.7071914	11.2740662	10.4772597
18	12.6592961	12.1599918	11.6895869	10.8276035
19	13.1339385	12.5932936	12.0853208	11.1581165
20	13.5903253	13.0079365	12.4622103	11.4699213
21	14.0091589	13.4047239	12.8211527	11.7640767
22	14.4511142	13.7844248	13.1630026	12.0415818
23	14.8568405	14.1477749	13.4885739	12.3033790
24	15.2469619	14.4954784	13.7986418	12.5503576
25	15.6220787	14.8282089	14.0939445	12.7833562

TABLE V.

The preſent Worth of one Pound, Annuity, Compound-Intereſt.

Years	2 *per C.*	2½ *per C.*	3 *per C.*	3½ *per C.*
26	20.1210358	18.9506111	17.8768420	16.8903523
27	20.7068978	19.4640109	18.3270315	17.2853645
28	21.2812724	19.9648887	18.7641082	17.6670188
29	21.8443847	20.4535499	19.1884546	18.0357670
30	22.3964556	20.9302926	19.6004413	18.3920454
31	22.9377015	21.3954074	20.0004285	18.7362758
32	23.4683348	21.8491780	20.3887655	19.0688656
33	23.9885636	22.2918809	20.7657918	19.3902082
34	24.4985917	22.7237863	21.1318367	19.7006842
35	24.9986193	23.1451573	21.4872200	20.0006612
36	25.4888425	23.5562511	21.8322525	20.2904938
37	25.9694534	23.9573181	22.1672354	20.5705254
38	26.4406406	24.3486030	22.4924616	20.8410874
39	26.9025888	24.7303444	22.8082151	21.1024999
40	27.3554792	25.1027751	23.1147719	21.3550723
41	27.7994895	25.4661220	23.4123999	21.5991037
42	28.2347936	25.8206068	23.7013592	21.8348828
43	28.6615623	26.1664457	23.9819021	22.0626887
44	29.0799631	26.5038495	24.2542739	22.2827910
45	29.4901599	26.8330239	24.5187125	22.4954503
46	29.8923136	27.1541696	24.7754490	22.7009181
47	30.2865820	27.4674826	25.0247078	22.8994378
48	30.6731196	27.7731537	25.2667066	23.0912443
49	31.0520780	28.0713695	25.5016569	23.2765645
50	31.4236059	28.3623117	25.7297640	23.4556179

TABLE

TABLE V.

present Worth of one Pound, Annuity, Compound-Intereſt.

Years.	4 *per C.*	$4\frac{1}{2}$ *per C.*	5 *per C.*	6 *per C.*
26	15.9827678	15.1466115	14.3751853	13.0031663
27	16.3295844	15.4513028	14.6430336	13.2105342
28	16.6630618	15.7428735	14.8981272	13.4061644
29	16.9837132	16.0218885	15.1410735	13.5907211
30	17.2920318	16.2888885	15.3724510	13.7648312
31	17.5884921	16.5443909	15.5928104	13.9290861
32	17.8735500	16.7888909	15.8026766	14.0840435
33	18.1476441	17.0228621	16.0025491	14.2302297
34	18.4111962	17.2467580	16.1929039	14.3681412
35	18.6646116	17.4610124	16.3741942	14.4982465
36	18.9082803	17.6660406	16.5468516	14.6209872
37	19.1425771	17.8622398	16.7112872	14.7367804
38	19.3678625	18.0499902	16.8678926	14.8460192
39	19.5844831	18.2296557	17.0170406	14.9490747
40	19.7927721	18.4015844	17.1590862	15.0462969
41	19.9930500	18.5661095	17.2943678	15.1380160
42	20.1856250	18.7235498	17.4232074	15.2245434
43	20.3707931	18.8742103	17.5459118	15.3061730
44	20.5488395	19.0183831	17.6627732	15.3831821
45	20.7200378	19.1563474	17.7740697	15.4558321
46	20.8846517	19.2883707	17.8800663	15.5243699
47	21.0429342	19.4147088	17.9610155	15.5890282
48	21.1951289	19.5356066	18.0771576	15.6500266
49	21.3414700	19.6512981	18.1687215	15.7075723
50	21.4821826	19.7620078	18.2559253	15.7618610

423. By Theorem II. (407.) a very uſeful Queſtion is reſolved, *viz.*

QUESTION II.

What Annuity, to continue ſeven Years, may be purchaſed for 120*l.* 5*s.* at 6 *per Cent.* Compound Intereſt.

Here is given $\left\{\begin{array}{l} P = 120,25 \\ R = 1,06 \\ t = 7 \end{array}\right\}$ to find U, the Annuity.

Involve the Rate - - $R = 1,06 = 0,025305$
To the Index of its Power (*viz.* $= t$) — 7

The Power of R will be $R^7 = 1,50361 = 0,177135$
Which multiply by the preſent Worth - - - $P = 120,25 = 2,080084$

The Product is - - $PR^t = 180,8087 = 2,257219$
Multiply that by the Rate - - $R = 1,06 = 0,025305$

That Product is - $PR^t \times R = 191,65722 = 2,282524$
From which ſubtract - $PR^t = 180,8087$
There remains the Dividend - $10,84852 = PR^t \times R - PR^t$.

Divide therefore $PR^t \times R - PR^t = 10,84852 = 1,035369$
By the Power of R leſs 1 $= R^t - 1 = 0,50361 = 9,702113$

The Quotient is the Annuity $U = 21,54057 = 1,333256$

The Annuity ſought therefore is $21,54057$*l.* = 21*l.* 10*s.* $9\frac{1}{2}$*d.*

424. But we ſhall next ſee, with how much greater Eaſe theſe Queſtions are ſolved by a proper Table; the Conſtruction of which is as follows.

The

The Conſtruction of TABLE VI.

This Table is made from Theorem II. (407.) by putting $P = 1l.$ which then is reduced to this Form, $RR^t - R^t = UR^t - U$; whence (at five *per Cent.*) it will be $,05\,R^t = UR^t - U$; conſequently $\frac{,05\,R^t}{R^t - 1} = U$, the Annuity required. But this being juſt the Reverſe of $\frac{R^t - 1}{,05\,R^t}$, which make the Numbers of Table V. it is plain, theſe two Theorems, which conſtitute the Numbers of Table V. and VI. multiplied together can make but 1, that is $\frac{,05\,R^t}{R^t - 1} \times \frac{R^t - 1}{,05\,R^t} = 1.$

425. Hence then, if the Numbers of Table V. be made Diviſors, and Unity, or 1, the conſtant Dividend, the Quotients ſhall be the Numbers which conſtitute the 6th Table, at 5 *per Cent.* and after the ſame Manner for any other Rate of Intereſt.

Examples at 5 per Cent.

The Numbers of Table V. at 5 *per C.* are	By which divide Unity, the Quotients will be	The Annuit. 1*l.* will purchaſe for	Years, *&c.*
,9523809	1,05		1
1,8594103	,5378049		2
2,7232480	,3637734		3
3,5459505	,2787437		4
4,3294767	,2277916		5

426. The Uſe of the Table in ſolving the above Queſtion, is ſhewn in three Lines only.

The tabular Number for the Time and Rate is -	0,179135
Which multiplied by the propoſed Sum - -	120,25
Gives the Annuity to be purchaſed, *viz.* -	£. 21,54098

TABLE VI.

Of COMPOUND INTEREST.

427. *The* Annuity *which* one Pound *will purcha*
for any Number of Years, *at the Rates of* 2, 2
3, 3½, 4, 4½, 5, *and* 6, per Cent. per Annum

Years.	2 *per C.*	2½ *per C.*	3 *per C.*	3½ *per C.*
1	1.0200000	1.0250000	1.0300000	1.0350000
2	.5150495	.5188272	.5226108	.5264005
3	.3467547	.3501372	.3535304	.3569342
4	.2626238	.2658179	.2690271	.2722511
5	.2121584	.2152469	.2183546	.2214814
6	.1785258	.1815499	.1845975	.1876682
7	.1545120	.1574954	.1605064	.1635445
8	.1365098	.1394674	.1424564	.1454767
9	.1225154	.1254569	.1284339	.1314460
10	.1113265	.1142588	.1172305	.1202414
11	.1021779	.1051060	.1080775	.1110920
12	.0945596	.0974871	.1004621	.1034840
13	.0881183	.0910483	.0940295	.0970616
14	.0826020	.0855365	.0885263	.0915707
15	.0778255	.0807665	.0837666	.0868251
16	.0736501	.0765990	.0796109	.0826848
17	.0699698	.0729278	.0759525	.0790431
18	.0667021	.0696701	.0727087	.0758168
19	.0637818	.0667606	.0698139	.0729403
20	.0611567	.0641471	.0672157	.0703610
21	.0587847	.0617873	.0648718	.0680366
22	.0566314	.0596466	.0627474	.0659321
23	.0546681	.0576964	.0608139	.0640188
24	.0528511	.0559128	.0590474	.0622728
25	.0512204	.0542759	.0574279	.0606740

TABLE

TABLE VI.

The Annuity which one Pound will purchaſe, Compound Intereſt.

Years.	4 *per Cent.*	4½ *perCent.*	5 *per Cent.*	6 *per Cent.*
1	1.0400000	1.0450000	1.0500000	[illegible].0600000
2	.5301961	.5339976	.5378049	.5454369
3	.3603485	.3637734	.3672086	.3741098
4	.2754901	.2787437	.2820118	.2885915
5	.2246271	.2277916	.2309748	.2373964
6	.1907619	.1938784	.1970157	.2033626
7	.1666096	.1097015	.1728198	.1791350
8	.1485279	.1516097	.1547218	.1610359
9	.1344930	.1375745	.1406901	.1470222
10	.1232909	.1263788	.1295046	.1358680
11	.1141490	.1172482	.1203890	.1267929
12	.1065522	.1096662	.1128254	.1192770
13	.1001437	.1032754	.1064558	.1129601
14	.0946690	.0978203	.1010240	.1275849
15	.0899411	.0931138	.0963423	.1029628
16	.0858200	.0890154	.0922699	.0989521
17	.0821985	.0854176	.0886991	.0954448
18	.0789933	.0822369	.0855462	.0923565
19	.0761386	.0794073	.0827450	.0896209
20	.0735818	.0768761	.0802426	.0871846
21	.0712801	.0746006	.0779961	.0850046
22	.0691988	.0725457	.0759705	.0830456
23	.0673091	.0706825	.0741368	.0812785
24	.0655868	.0689870	.0724709	.0796790
25	.0640121	.0674390	.0709545	.0782267

TABLE VI.

The Annuity which one Pound will purchase Compound Interest.

Years.	*2 per Cent.*	*2½ perCent.*	*3 per Cent.*	*3½ perCent.*
26	0.0496992	0.0527688	0.0559383	0.0592054
27	0.0482931	0.0513769	0.0545642	0.0578524
28	0.0469897	0.0500879	0.0532932	0.0566027
29	0.0457784	0.0488913	0.0521147	0.0554454
30	0.0446499	0.0477776	0.0510193	0.0543713
31	0.0435964	0.0467390	0.0499989	0.0533724
32	0.0426106	0.0457683	0.0490466	0.0524415
33	0.0416865	0.0448594	0.0481561	0.0515724
34	0.0408187	0.0440068	0.047322	0.0507597
35	0.0400022	0.0432056	0.0465393	0.0499984
36	0.0392329	0.0424516	0.0458038	0.0492842
37	0.0385068	0.0417409	0.0451116	0.0486133
38	0.0378206	0.0410701	0.0444593	0.0479821
39	0.0371711	0.0404362	0.0438439	0.047 878
40	0.0365558	0.0398362	0.0432624	0.0468273
41	0.0359719	0.0392679	0.0427124	0.0462982
42	0.0354173	0.0387288	0.0421917	0.0457983
43	0.0348899	0.0382169	0.0416981	0.0453254
44	0.0343879	0.0377304	0.0412299	0.0448777
45	0.0339096	0.0372675	0.0407852	0.0444534
46	0.0334534	0.0368268	0.0403625	0.0440511
47	0.0330179	0.0364067	0.0399605	0.0436692
48	0.0326018	0.0360060	0.0395778	0.0433065
49	0.0322040	0.0356235	0.0392131	0.0429617
50	0.0318032	0.0352581	0.0388655	0.0426337

TABLE VI.

he Annuity which one Pound will purchaſe Compound-Intereſt.

Years	4 *per* C.	4 ½ *per* C.	5 *per* C.	6 *per* C.
26	0.0625674	0.0660214	0.0695643	0.0769044
27	0.0612385	0.0647195	0.0682919	0.0756972
28	0.0600130	0.0635208	0.0671225	0.0745926
29	0.0588799	0.0624146	0.0660455	0.0735796
30	0.0578301	0.0613915	0.0650514	0.0726489
31	0.0568554	0.0604435	0.0641321	0.0717522
32	0.0559486	0.0595632	0.0632804	0.0710023
33	0.0551036	0.0587445	0.0624900	0.0702729
34	0.0543148	0.0579819	0.0617554	0.0695984
35	0.0535773	0.0572705	0.0610717	0.0689739
36	0.0528869	0.0566058	0.0604345	0.0683948
37	0.0522396	0.0559840	0.0598398	0.0678574
38	0.0516319	0.0554017	0.0592842	0.0673581
39	0.0510608	0.0548557	0.0587646	0.0668938
40	0.0505235	0.0543431	0.0582782	0.0664915
41	0.0500174	0.0538616	0.0578223	0.0660589
42	0.0495402	0.0534087	0.0573947	0.0656834
43	0.0490899	0.0529824	0.0569933	0.0653331
44	0.0486645	0.0525807	0.0566163	0.0650061
45	0.0482625	0.0522020	0.0562617	0.0647005
46	0.0478821	0.0518447	0.0559282	0.0644149
47	0.0475219	0.0515073	0.0556142	0.0641477
48	0.0471807	0.0511886	0.0553184	0.0638977
49	0.0468571	0.0508872	0.0550397	0.0636636
50	0.0465502	0.0506021	0.0547767	0.0634443

428. By Theorem III. (417.) you find the Time for which a propoſed Annuity may be purchaſed, for a given Sum, at a given Rate of Intereſt, as in the following Examples.

QUESTION III.

For what Time will 167*l.* 9*s.* 5*d.* purchaſe an Annuity of 30*l.* *per Ann.* at 6 *per Cent.* Compound Intereſt?

$$\text{Here is given}\left\{\begin{array}{l}P = 167{,}4716\\ U = 30\\ R = 1{,}06\end{array}\right\}\text{to find } t\text{, } \textit{per} \text{ Theor. III.}$$

To the preſent Worth - - $P = 167{,}4716$
Add the Annuity - - $U = 30$

The Sum is - - $P + U = 197{,}4716$
Then multiply the preſent Worth $P = 167{,}4716 = 2{,}223928$
By the Rate - - - - $R = 1{,}06 = 0{,}025305$

The Product is - - $PR = 177{,}5199 = 2{,}249233$
Which ſubtract from Sum of $P + U = 197{,}4716$

The Remainder is $P + U - PR = 19{,}9517$, the Diviſor.

Then divide the Annuity - - $U = 30 = 1{,}477121$
By the Diviſor - $P + U - PR = 19{,}9517 = 1{,}299986$

The Quotient is - - $R^t = 1{,}50361 = 0{,}177135$

Then in Table II. under 6 *per Cent.* you find the Number 1,5036303, againſt which, in the Side is 7 Years, the Anſwer.

QUESTION IV.

429. Suppoſe I purchaſe an Annuity of 21*l.* 10*s.* 9½*d.* to continue 7 Years, for 120*l.* 5*s.* ready Money; at what Rate *per Cent.* Compound Intereſt, was the Purchaſe made?

$$\text{Here is given}\left\{\begin{array}{l}P = 120{,}25\\ U = 21{,}54057\\ t = 7\end{array}\right\}\text{to find } R\text{, } \textit{per} \text{ Theor. IV.}$$

Firſt, divide the Annuity - $U = 21{,}54057 = 1{,}333256$
By the preſent Worth - $P = 120{,}25 = 2{,}080084$

The Quotient is - - $\frac{U}{P} = 0{,}17915 = 9{,}253172$

Then multiply it into the given Power of the Rate, to which dd the Power, &c. as *per* Theorem; and you have this Equa-ion, *viz.* $0{,}17915\,R^7 + R^7 - R^8 = 0{,}17915$; or $1{,}17915$ $R^7 - R^8 = 0{,}17915$; whence R will be found (by Table II.) fter the ſame Manner as in (414.) to be 1,06, or that the Rate 6 *per Cent.*

CHAP. XX.

HEOREMS *reſolving all Queſtions relating to the purchaſing of* Freehold, *or* Real Eſtates, *at* Compound-Intereſt.

30. TO purchaſe a Freehold Eſtate is evidently nothing more than to find the preſent Worth of an Annuity, o continue for ever; and conſequently the Theor. (416.) $PR^t = \frac{UR^t - U}{R-1}$, will alſo ſerve our Purpoſe here, if we make e Index (t) infinite (as in this Caſe the Time really is;) for en (ſince a finite Quantity, U, ſubtracted from an infinite e, makes no Alteration) we have

$$PR^t = \frac{UR^t}{R-1}; \text{ or, } \overline{PR - P} \times R^t = UR^t.$$

That is, $PR - P = U$, Theorem I.

And $\frac{U}{R-1} = P$, Theorem II.

$$\text{And } \frac{P+U}{P} = R, \text{ Theorem III.}$$

By theſe Theorems, the following uſeful Queſtions ar ſolved.

QUESTION I.

431. Suppoſe a Freehold Eſtate of 25*l.* *per Ann.* were to b ſold, what is the Worth, allowing 5*l.* 10*s.* *per Cent.* &c. Com pound-Intereſt to the Buyer?

Here is given $\left\{\begin{matrix} U = 25 \\ R = 1{,}055 \end{matrix}\right\}$ To find P, *per* Theorem II.

Divide the annual Rent - - $U = 25 = 1{,}39794$
By the Rate leſs Unity - $R - 1 = 0{,}055 = 8{,}74036$

The Quotient is the Worth - $P = 454{,}5 = 2{,}65757$

The Value of that Eſtate therefore is 454*l.* 10*s.* 10¾*d.* Q. E.

QUESTION II.

432. Suppoſe a Perſon would lay out 410*l.* 13*s.* 4*d.* on Freehold Eſtate, and ſo as to be allowed 6*l.* *per Cent.* for hi Money, Compound-Intereſt; what muſt be the annual Rent ſuch an Eſtate?

Here is given $\left\{\begin{matrix} P = 410{,}6 \\ R = 1{,}06 \end{matrix}\right\}$ To find U, *per* Theorem I.

Multiply the preſent Worth - $P = 410{,}6 = 2{,}6197$
By the Rate - - $R = 1{,}06 = 0{,}0253$

The Product is - - $PR = 441{,}0 = 2{,}6450$
From which ſubtract the Worth $P = 410{,}6$

There remains the annual Rent. The Anſwer. $\Big\}$ $U = 25$*l.* *per Ann.*

QUES

QUESTION III.

433. Suppoſe one give 416*l.* 13*s.* 4*d.* for a Freehold Eſtate of 25*l.* *per Ann.* what Rate *per Cent.* Compound-Intereſt, has the Purchaſer for his Money?

Here is given $\left\{\begin{array}{l} P = 41\cancel{6},6 \\ U = 25 \end{array}\right\}$ To find R, *per* Theor. III.

To the preſent Worth - $P = 41\cancel{6},6$
Add the annual Rent - $U = 25,0$

Divide their Sum - $P + U = 441,\cancel{6} = 2,645094$
By the preſent Worth - $P = 41\cancel{6},6 = 2,619789$

The Quotient is the Rate ſought $R = 1,06 = 0,025305$

Then ſay, As 1*l.* : ,06*l.* :: 100*l.* : 6*l.* *per Cent.* the Anſwer.

434. That nothing may be wanting to facilitate the Computations of Simple and Compound-Intereſt, I ſhall here ſubjoin two Tables, to expreſs the Time in Parts of a Year, without any Trouble. One of which will ſhew the Number of Days from the Beginning of the Year to any Day of a Month propoſed; and the other will ſhew what Decimal Part of a Year they make. The firſt here follows, with its Uſe.

435. *A* TABLE *of* DAYS *for any given Time leſs than a Year.*

Days.	*January*	*February*	*March*	*April*	*May*	*June*	*July*	*August*	*September*	*October*	*November*	*December*
1	1	32	60	91	121	152	182	213	244	274	305	335
2	2	33	61	92	122	153	183	214	245	275	306	336
3	3	34	62	93	123	154	184	215	246	276	307	337
4	4	35	63	94	124	155	185	216	247	277	308	338
5	5	36	64	95	125	156	186	217	248	278	309	339
6	6	37	65	96	126	157	187	218	249	279	310	340
7	7	38	66	97	127	158	188	219	250	280	311	341
8	8	39	67	98	128	159	189	220	251	281	312	342
9	9	40	68	99	129	160	190	221	252	282	313	343
10	10	41	69	100	130	161	191	222	253	283	314	344
11	11	42	70	101	131	162	192	223	254	284	315	345
12	12	43	71	102	132	163	193	224	255	285	316	346
13	13	44	72	103	133	164	194	225	256	286	317	347
14	14	45	73	104	134	165	195	226	257	287	318	348
15	15	46	74	105	135	166	196	227	258	288	319	349
16	16	47	75	106	136	167	197	228	259	289	320	350
17	17	48	76	107	137	168	198	229	260	290	321	451
18	18	49	77	108	138	169	199	230	261	291	322	352
19	19	50	78	109	139	170	200	231	262	292	823	353
20	20	51	79	110	140	171	201	232	263	293	324	354
21	21	52	80	111	141	172	202	233	264	294	325	355
22	22	53	81	112	142	173	203	234	265	295	326	356
23	23	54	82	113	143	174	204	235	266	296	327	357
24	24	55	83	114	144	175	205	236	267	297	328	358
25	25	56	84	115	145	176	206	237	268	298	329	359
26	26	57	85	116	146	177	207	238	269	299	330	360
27	27	58	86	117	147	178	208	239	270	300	331	361
28	28	59	87	118	148	179	209	240	271	301	332	362
29	29		88	119	149	180	210	241	272	302	333	363
30	30		89	120	150	181	211	242	273	303	334	364
31	31		90		151		212	243		304		365

The

The USE *of the* TABLE.

436. First; To know the Number of Days from the Begin-
ing of the Year, to any given Day of any Month.

This is obtained by Inſpection only; thus from *January* the
ſt, to *September* the 7th, is 250 Days; to *November* the 27th
re 331, *&c.*

Secondly, To know what is the Number of Days from any
iven Day of any Month, to the End of the Year.

Suppoſe *September* 7, then from - - 365
Subtract the Number anſwering to *September* 7 250

There remains the Number of Days ſought, *viz.* 115 Days.

Thirdly, To find the Number of Days between the given Day
f any one Month, and any given Day of any other Month,
the ſame Year.

For Inſtance, To know how many Days there are between
pril the 17th and *October* 23.

Thus, from the Number anſwering to *October* 23 - 296
Subtract that anſwering to *April* 17. - - 107

The Remainder is the Number of Days ſought. 189

Fourthly, To find the Number of Days from any given Day
any Month in one Year, to any given Day of any Month
the next Year.

How many Days is it from *September* the 7th in one Year,
April the 19th in the next?

From

From the Days of a whole Year - -	365
Subtract the Number to *September* 7 -	250
Remains the Number to the End of the Year	115
To which add the Number to *April* 19 -	109
The Sum is the Number of Days required.	224

And thus is the Number of Days readily found for any
terval of Time given, in the ſame Year compleatly; or wh
is Part of one, or Part of another Year.

437. Having then the Number of Days, it is eaſy to
what Decimal Part of the Year they make, in the follow
Table; and having found that, you have the Symbols, T
in the foregoing Theorems, repreſenting any Part of a Ye

438. *A* Decimal Table *of Days and Months in a Year.*

		10 Days	20 Days	30 Days	40 Days	50 Days	60 Days	70 Days	80 Days	90 Days	
0		.027397	.054795	.082192	.109589	.136986	.164384	.191781	.219178	.246575	0
1	.002740	.030137	.057534	.084932	.112329	.139726	.167123	.194521	.221918	.249315	1
2	.005479	.032877	.060274	.087671	.115068	.142466	.169863	.197260	.224658	.252055	2
3	.008219	.035616	.063014	.090411	.117808	.145205	.172603	.2	.227397	.254795	3
4	.010959	.038356	.065753	.093151	.120548	.147945	.175342	.202740	.230137	.257534	4
5	.013699	.041096	.068493	.095890	.123288	.150685	.178082	.205479	.232877	.260274	5
6	.016438	.043836	.071233	.098630	.126027	.153425	.180822	.208219	.235616	.263014	6
7	.019178	.046575	.073973	.101370	.128767	.156164	.183562	.210959	.238356	.265753	7
8	.021918	.049315	.076712	.104110	.131507	.158904	.186301	.213699	.241096	.268493	8
9	.024658	.052055	.079452	.106849	.134247	.161644	.189041	.216438	.243836	.271233	9
	Months.	Decim.	Months.	Decim.	Months.	Decim.	Months.	Decim.	100	.273973	
	1	.083333	4	.333333	7	.583333	10	.833333	200	.547945	
	2	.166667	5	.416667	8	.666667	11	.916667	300	.821918	
	3	.25	6	.5	9	.75	12	1.	365	1.	

The Uſe of the foregoing Table.

439. The decimal Part of a Year for Days under 10, is contained in the firſt Column; thus, for ſeven Days it is 0.019178.

For any Number of Days from 10 to 100, look for the even Tens on the Top, and the Units on the Side of the Table; and where they meet in the Table is the Number expreſſing the decimal Part of the Year; thus the Decimal of 47 Days is found under 40 on the Top and againſt ſeven on the Side, to be 0.128767. The Decimal of 73 Days is 0.2.

In the lower Part of the Table are the Decimals of a Year for Months; and 100, 200 or 300 Days. So that for three Calendar Months and 27 Days, the Decimal of a Year is 0,25 + 0.073973 = 0.323973. Or for 275 Days, it is 0.547945 + 0.205479 = 0.7503424.

440. As it is impoſſible to avoid the Uſe of Decimals in *Money* as well as in *Time*, in the Computations of Intereſt, as appears in all the preceeding Caſes and Examples; therefore it is expedient to ſubjoin the following Table of the dicimal Parts of a Pound Sterling, anſwering to any Number of Shillings, Pence, and Fathings, propoſed.

Thus, ſuppoſe it is required to expreſs 57 *l.* 17 *s.* 9 *d.* ¾ in Decimals. You take from the Table the correſponding Decimals for the reſpective Parts, and then add them together, as below.

	£.
The integral Part	57.
againſt { 17 *s.* is	0.85
9 *d.* ¾	0.040625
The Sum in Decimals	57.890625

441. Again, by the ſame Table any given decimal Sum may be reſolved into the common Species of Money; as the Sum 67,185416.

£.
The integral Part is 67.
From the Decimal 0,18541$\not{6}$
Take the next leſs for 3*s.* = 0,15
The Remainder anſwers to 8 $\frac{1}{2}$ = 0,03541$\not{6}$

Therefore the Sum £. 67,18541$\not{6}$ = 67 *l.* 3 *s.* 8 *d.* $\frac{1}{2}$.

442. *A Table of the Decimal Parts of a Pound Sterling.*

A DECIMAL TABLE
Of Money; *one* Pound *the Integer.*

S.	*Decimals.*	*P. q.*	*Decimals.*	*P. q.*	*Decimals.*
1	,05	0 $\frac{1}{8}$	,0005208	6,0	,025
2	,1	0 $\frac{1}{4}$	,001041$\not{6}$	6 $\frac{1}{4}$	,026041$\not{6}$
3	,15	0 $\frac{1}{2}$	,00208$\not{3}$	6 $\frac{1}{2}$	,027008$\not{3}$
4	,2	0 $\frac{3}{4}$	,003125	6 $\frac{3}{4}$	,028125
5	,25	1,0	,0041$\not{6}$	7,0	,0291$\not{6}$6
6	,3	1 $\frac{1}{4}$	,0052	7 $\frac{1}{4}$	,030208$\not{3}$
7	,35	1 $\frac{1}{2}$	,00625	7 $\frac{1}{2}$	,03125
8	,4	1 $\frac{3}{4}$	,00729	7 $\frac{3}{4}$	,032291$\not{6}$
9	,45	2,0	,008$\not{3}$3	8,0	,0$\not{3}$3333
10	,5	2 $\frac{1}{4}$	,009375	8 $\frac{1}{4}$	,034375
11	,55	2 $\frac{1}{2}$	,01041$\not{6}$	8 $\frac{1}{2}$	,03541$\not{6}$
12	,6	2 $\frac{3}{4}$	,011458$\not{3}$	8 $\frac{3}{4}$	,036458$\not{3}$
13	,65	3,0	,0125	9,0	,0375
14	,7	3 $\frac{1}{4}$	,013541$\not{6}$	9 $\frac{1}{4}$	,038541$\not{6}$
15	,75	3 $\frac{1}{2}$	,01458$\not{3}$	9 $\frac{1}{2}$	,03958$\not{3}$
16	,8	3 $\frac{3}{4}$	,015625	9 $\frac{3}{4}$	,040625
17	,85	4,0	,01$\not{6}$666	10,0	,041$\not{6}$66
18	,9	4 $\frac{1}{4}$	,017708$\not{3}$	10 $\frac{1}{4}$	,042708$\not{3}$
19	,95	4 $\frac{1}{2}$	,01875	10 $\frac{1}{2}$	,04375
		4 $\frac{3}{4}$	,019791$\not{6}$	10 $\frac{3}{4}$	,044791$\not{6}$
		5,0	,0208$\not{3}$	11,0	,0458$\not{3}$3
		5 $\frac{1}{4}$	,021875	11 $\frac{1}{4}$	,046875
		5 $\frac{1}{2}$	,02291$\not{6}$	11 $\frac{1}{2}$	,04791$\not{6}$
		5 $\frac{3}{4}$	,023958$\not{3}$	11 $\frac{3}{4}$	,048958$\not{3}$

443. An Example in Simple and Compound Intereſt, wil make the whole Matter eaſy and conſpicuous.

EXAMPLE I.

What will 65*l.* amount to, being lent from *March* the 7th to *November* the 3d, at 5*l.* *per Cent. per Annum*, Simple-In- tereſt ?

From *March* the 7th to *November* the 3d are 241 Days, b (435); thoſe make 8 Months, 2 Weeks, and three Days, = 0,660273 Decimals of a Year, by (438.) Then by Theorem I (270)

Multiply the Time - - - $T = 0,66027$
By the *Ratio* of the Rate - - $R = 0,0$

And that Product - - - $TR = 0,0330136$
Multiply by the Principal - - $P =$

The Product is - - - $TRP = 2,1458872$
To which add the Principal - - $P = 65$

The Sum is the Amount ſought - $= 67,1458$*l.*

EXAMPLE II.

What is the Amount thereof at Compound-Intereſt, th Rate and Time being the ſame?

The Logarithm of the Rate - $R = 1,05 = 0,021189$
Multiply by the Time - $t = ,660$

The Product is the Logar. of $R^t = R^{0.6603} = 0,013991$
To which add the Log. of the Prin. $P = 63 = 1,812913$

The Sum is the Log. of Amount $A = 67,1281 = 1,826904$

And thus the Theorems ſerve to anſwer Queſtions, when th Time is only Part of a Year, as well as when complete Year

444. I ſhall now add a few Queſtions of a *mix'd N ture*, and which frequently happen, in order to ſhew the mo extenſive Uſe of the Tables.

QUES

QUESTION I.

Suppoſe I have 790*l.* to be paid me within ſeven Years, in this Manner; at the End of the firſt Year 90*l.* of two Years 100*l.* of four Years 200*l.* and of ſeven Years 400*l.* Query, what the preſent Worth of thoſe ſeveral Payments is in ready Money, allowing $4\frac{1}{2}$ *per Cent.* Compound Intereſt?

In Table III. the preſent Worth of 1*l.* at $4\frac{1}{2}$ *per Cent* due at the End of one Year, is - - } 0.9569378
Which multiply by the Principal - - - - 90

The Product is the preſent Worth of 90*l.* = 86.124402
Thus the preſent Worth of 100*l.* due at the End of two Years, is found - - } = 91.57299
Alſo, of 200*l.* at the End of four Years = 167.71226
And of 400*l.* at the End of ſeven Years = 293.93140

The Sum of all theſe is - - - - *l.* 639.341052

Which anſwers the Queſtion, *viz.* 639*l.* 6*s.* $9\frac{3}{4}$*d.*

QUESTION II.

445. *A* owes to *B* 455*l.* to be paid in 14 Years, *viz.* at the End of every two Years 65*l.* But he would agree to pay him in ſeven Years, by *equal Payments* each Year; which *B* agrees to, and at the Rate of 6 *per Cent. Compound Intereſt.* Query, what the *annual Payment* muſt be?

1. Find the *preſent Worth* (by Table III.) of the ſeven Payments which were at firſt to be made, as *per Queſt.* I. which you will find to be 293*l.* 5*s.* 2*d.*
2. Then find (by Table VI.) what *Annuity*, to continue ſeven Years at the given Rate, 293*l.* 5*s.* 2*d.* will purchaſe; which you will find to be 52*l.* 10*s.* 8*d.* and is the Anſwer to the Queſtion.

QUESTION III.

446. *A* has a Term of ſeven Years in an Eſtate of 35*l.* *per Annum.* *B* has a Term of 14 Years in the ſame Eſtate in *Reverſion* after the ſeven Years; and *C* has a farther Term of

20 Years in *Reverſion* after the 21 Years. Query the preſent Values of the *ſeveral Terms*, at the Rate of 5 *per Cent. per Annum?*

By *Table* V. the *preſent Value* of 35 *l. per Annum*, may be

	l.	*s.*	*d.*
found, for 41 Years, to be —— ——	605	6	$0\frac{1}{2}$
for 21 Years, to be —— — ——	448	14	$9\frac{1}{2}$
for 7 Years, to be ——	202	10	$5\frac{1}{2}$

Which ſubtract from each other, and it will appear,

	l.	*s.*	*d.*
That the *preſent Value* of *A*'s Term is	202	10	$5\frac{1}{2}$
of *B*'s Term	246	4	4
of *C*'s Term	156	11	3
For theſe *Values* anſwer the Queſtion *l.*	605	6	$0\frac{1}{2}$

QUESTION IV.

447. Which is moſt advantagious a *Term* of 15 Years in an Eſtate of 100 *l. per Ann.* or the *Reverſion* of ſuch an Eſtate for ever after the Expiration of the ſaid 15 Years; computing at the Rate of 5 *per Cent. per Ann.* Compound Intereſt?

	£.
An Eſtate of 100*l. per Ann.* in Fee Simple, at 5 *per Cent.* is worth - - - -	2000
In Table V. the preſent Value of the ſame Eſtate at the ſame Rate, for 15 Years, is	1037,9658
The Difference is	962,0342

Now this Difference being the Value of the Reverſion, it appears that the firſt Term of 15 Years is better than the Reverſion for ever afterwards by 75,9316*l.* = 75 *l.* 18 *s.* $7\frac{1}{2}$ *d.* Anſwer.

QUESTION V.

448. A Perſon having 12 Years to come in a Leaſe of an Eſtate of 60*l. per Ann.* for 40 Years, would know what preſent Money he muſt pay in order to renew or complete the Leaſe, by adding

dding 28 Years thereto, computing at 6 *per Cent*. Compound ntereſt?

	£.
y Table V. the preſent Value of 1*l.* *per Ann.* at 6 *per Cent.* for 40 Years, is	15.046297
y the ſame Table the Value of 1*l.* *per Ann.* at that Rate, for 12 Years to come, is	8.383844
The Difference is	6.662453
Which multiplied by	60
The Product is the Anſwer, *viz.*	399.747180

In Money, 399*l.* 14*s.* 11*d.*

QUESTION VI.

449. *A.* gives 1550 *l.* for an Annuity of 100 *l.* *per Annum* for o Years. *B.* puts 1550*l.* out at Intereſt. It is required to now which will amount to the greateſt Sum at the End of he 50 Years, at the Rate of 6*l.* *per Cent.* &c. Compound ntereſt?

	£.
By Table IV. the Amount of 100 *l.* An-uity, in 50 Years at 6 *per Cent.* may be ound to be	29033,59046
By Table II. it may be found, that the Amount of 1550 *l.* for that Time and Rate will be	28551,23885
Hence *A's Annuity* is more than *B*'s 1550 *l.* by	482,35161

t the End of 50 Years. The preſent Value of which Diffe-ence is found, by Table III. to be 26 *l.* 3 *s.* 8½ *d.* and ſo much was *A*'s Caſe *better* than *B*'s.

QUESTION VII.

450. What Annuity, to continue 14 Years, may be purchaſed with 1000 *l.* due at the End of five Years; the Annuity o commence preſently, at 5*l.* *per Cent.*?

By

	£.
By Table III. the preſent Worth of 1000 *l.* due five Years hence at 5 *per Cent.* may be found	= 783,526
By Table VI. it may be found, that the Annuity which 783,5262 *l.* will purchaſe for 14 Years, at the Rate of 5 *per Cent.* is ——	= 79,151

In Money, 79 *l.* 3 *s.* 0¼ *d.* *per Annum*, the Anſwer.

QUESTION VIII.

451. For a Leaſe of certain Profits for ſeven Years, *A*, makes *two* Offers, either to pay 150 *l.* as a *Fine*, and 300 *l.* *per Annum*; or 1700 *l.* *Fine*, without any *Rent*. *B*, bids 650 *l.* *Fine*, and 200 *l.* *per Annum*. And *C*, offers 200 *l.* *Fine*, and 405 *l.* *per Annum*. Query which is the *beſt* Offer, and what the *Difference*, computing at 5 *l.* *per Cent.* &c. *Compound Intereſt*?

	£.
1. By Table II. the Amount of 150 *l.* in ſeven Years, at 5 *per Cent.* may be found to be	211,0659
By Table IV. the Amount of 300 *l.* *per Ann.* in ſeven Years at the given Rate may be found	242,6025
Therefore *A*'s Offer, at the End of ſeven Years would be ——— ———	2453,6684
2. By Table II. the Amount of 1700*l.* in ſeven Years. (*A*'s ſecond Offer) at the ſaid Rate, is found to be - - - - -	2392,0802
3. By Table II. the Amount of 650*l.* in ſeven Years, at the given Rate, will be found to be	914,6189
By Table IV. the Amount of 200*l.* *per Ann.* in 7 Years, at that Rate, will be found to be	1628,4016
Therefore *B*'s Offer will, in 7 Years, amount to	2543,0205
4. By Table II. the Amount of 200*l.* in ſeven Years, at the given Rate, will be found to be	281,4212
By Table IV. the Amount of 405*l.* *per Ann.* for the given Time and Rate, will be found to be	3297,5132
So that *C*'s Offer, in 7 Years, will amount to	3578,9344

The Amounts therefore of the ſaid Offers, at the End of the ſaid Term,, being thus known, the preſent Worth of the ſeveral Amounts, may be found by Table III. which are as follow.

	l.	s.	d.
The preſent Worth of *A*'s firſt Offer will be	1885	18	3
A's ſecond Offer	1700	00	0
B's Offer - -	1807	5	6
C's Offer - - -	2543	9	8

Therefore the preſent Worth of what *C*. offers is more than

	l.	s.	d.
A's firſt Offer, by	657	11	5
A's ſecond Offer, by	843	9	8
B's Offer, by -	736	4	2

Which fully anſwers the Queſtion.

N. B. This Queſtion might be more readily anſwered by finding the preſent Worths of the ſeveral offered Annuities (as *per* Table V.) and adding to them the ſeveral Fines, as the Reader may try at his Leiſure.

QUESTION IX.

452. What Annuity is ſufficient to pay off a Debt of 50 Millions in 30 Years, at 4*l.* *per Cent.* Compound Intereſt?

In Table IV. againſt 30 Years, under 4 *per C.* is 0,0578301
Which multiply by the Debt - - - 50000000

The Product is the Annuity ſought, *viz.* *l.* 2891505 *per Annum.*

So that ſuppoſing the National Debt to be 50 Millions, and he Intereſt paid to be two Millions *per Ann.* or 4*l.* *per Cent.* hen will a Sinking Fund of 891505*l.* *per Ann.* clear the whole Debt in 30 Years.

N. B. By this Example appears the Neceſſity of continuing he tabular Numbers to ſo many Places of Decimals.

QUESTION X.

453. Suppoſe one Farthing had been lent at Compound Intereſt at 5 *per Cent.* in the firſt Year of the Chriſtian Æra, or Birth of Chriſt, and ſo continued to the Year 1750; Query, the Amount thereof?

N. B. Having ſaid enough about the Uſe of the Tables, I here intend only to give the Reader a Hint of the ſurpriſing Nature of Numbers in geometrical Proportion. (See Theor. I. Art. 390.)

l.
Therefore, the Logarithm of the Rate 1,05 = 0,0211893
Multiplied by the Time - - - - - - 1750

The Product is - - - - - - 37,0812750
To which add the Logarithm of one Farthing, or the ,0010416 Part of a Pound, *viz.* - - - - = ,7,0177288

The Sum is the Log. of the Amount ſought = 34,099003

Now the Index of this Logarithm being 34, ſhews the Number of Figures, of which the Amount of one Farthing in th given Time doth conſiſt, to be 35, of which let it be ſufficient to expreſs the five firſt in Figures; the reſt in Cyphers; the will the ſaid Amount be

12561000000000000000000000000000000*l.*

Now the Value of a ſolid Body, perfectly ſpherical, who Diameter is 8000 *Engliſh* Miles, (which is ſomewhat bigge than the Diameter of the Globe of our Earth.) I ſay ſuch ſolid Body of fine Gold would be in Value about

2386600000000000000000000000*l.*

Now if from each of theſe great Numbers be cut off 2 Cyphers, the remaining Figures will be 125610000000 in th Amount of the Farthing; and 23866 in the Value of th Globe of Gold. But 23866) 125610000000 (= 526000 nearly.

Hen

Hence it appears, that one ſingle Farthing put out to Uſe in the Manner aforeſaid, would amount to more in Value than five Millions and two hundred and ſixty thouſand Globes of fine ſolid Gold, each bigger than the Globe of the Earth! A ſtrange and ſurpriſing, but no leſs certain Truth! And this immenſe Amount would be greatly increaſed by inlarging the Rate of Intereſt.

454. As it may be ſome Curioſity to the Reader to know when theſe wonderful Queſtions firſt came in Uſe, and on what Occaſion, I ſhall conclude this Subject with an Account thereof from the learned Dr. *Wallis*, and in his own Words, which he ſubjoins to his Solution of the Queſtion of *buying a Horſe by giving a Farthing fer the firſt Nail, and doubling the Sum for every Nail in his Four Shoes.* His Words are:

" The firſt Occaſion of which Queſtion, I believe to be what I have cited, *Cap.* 13. of my *Opus Arithmeticum*, from *Alſephad* (an *Arabic* Writer) in his *Commentaries* upon *Tograius's* Verſes: Namely, That one *Seſſa*, an *Indian*, having firſt found out the Game at *Cheſſe*, and ſhewed it to his Prince *Shebram:* The King, who was highly pleaſed with it, bid him aſk what he would for the Reward of his Invention; whereupon he aſked, that for the firſt little Square of the Cheſſe-board, he might have one Grain of Wheat given him; for the ſecond, two; and ſo on doubling continually, according to the Number of Squares in the Cheſſe-board, which was 64. And when the King, who intended to give a very noble Reward, was much diſpleaſed, that he had aſked ſo trifling a one; *Seſſa* declared, that he would be contented with this ſmall one. So the Reward he had fixed upon, was ordered to be given him: But the King was quickly aſtoniſhed, when he found that this would riſe to ſo vaſt a Quantity, that the whole Earth itſelf could not furniſh out ſo much Wheat. But how great the Number of theſe Grains is, may be found by doubling one continually 63 Times, ſo that we may get the Number that comes in the laſt Place; and then one Time more yet to have the Sum of all. For the Double of the laſt Term (leſs by one) is the Sum of all. Now this will be more expeditiouſly done by Logarithms, and accurately enough too for this Purpoſe. For the Log. of 2 (which is

0.3010300)

0.3010300) multiplied by 64, is 19.2659200; the abſolute Number agreeing to this, will be greater than 18446.00000.00000.00000 and leſs than 18447.00000.00000.00000."

Then by allowing 7680 Wheat Corns to a Statute Pint, and Wheat at 4 *s.* per Buſhel, the Reader may find, at his Leiſure, to what an immenſe Sum the Value of the above-mentioned Grains will ariſe.

CHAP. XXI.

The Valuation of Annuities upon Lives.

455. AS this is a very neceſſary Subject, we ought not t paſs over it in this Place; and tho' it has bee handled by learned Pens, yet, as ſome have given Rules an Solutions to Queſtions and Caſes of this Sort, without an Theory or Demonſtration at all; and others have given Theorems too tedious and perplexed; we ſhall here endeavour t ſteer a Courſe between both; and propoſe ſome of the moſt uſeful Caſes, with as plain a Rationale thereof as we can.

456. And it is firſt to be obſerved, that before the Value o any Annuity on a Life of any propoſed Age can be at all aſcertained, it will be previouſly neceſſary to conſider, how w are to eſtimate the Probability of the Continuance of a Li for any given Time; and it is evident this can be done n other Way but from Obſervations made from the *Bills of Mortality*, for a Series of Years together, and ſuch as are prop to the Country where thoſe Computations are to be mad For the *Bills of Mortality* at *Breſlaw* will by no Means fu the Meridian of *London*; and therefore we ſhall firſt exhibit Table of the Number of Perſons who have died in eve Year from 1728 to 1747 incluſive, or 20 Years, as taken fro the yearly Bills of Mortality of this Metropolis, for the Ag mentioned at the Top of the reſpective Colums.

457. T

457. The Bills of Mortality alſo mention the Number of Perſons born in each Year; and from a Mean of the Whole, or 1000 Perſons born, there die the Numbers mentioned in the firſt Table at the Bottom, for the ſeveral Intervals of Age, as here ſpecified; where 'tis obſervable, that the leaſt Number ie (*viz* 31 in 1000) between 10 and 20; and the greateſt Number (*viz.* 96 in 1000) between 40 and 50 Years of Age.

458. From the firſt Table the Second is conſtructed, where the Numbers of the ſeveral Intervals of Age are diſtributed into Numbers proper to the Years of thoſe Intervals; for Inſtance, between 10 and 20 Years there die 31 Perſons; but theſe in the following Table are divided and allotted to each ſingle Year as follows; 4, 4, 3, 2, 2, 2, 2, 3, 4, 5, the Sum of which makes 31. From hence the Probability of Living any propoſed Number of Years, at any Stage of Life, may be very eaſily deduced, as we ſhall ſhew.

459. TABLE

TABLE I.

459. *An Account of the Number of Perſons that Twenty Years, taken from the Yearly Bills of*

Years.	Under 2 Years of Age.	between 2 & 5	5 & 10	10 & 20	20 & 30	30 & 40
1728	9851	2407	1038	950	2254	2490
1729	10735	2516	1056	999	2371	2784
1730	10368	2448	1092	901	2048	2471
1731	9907	2096	932	806	1916	2351
1732	9502	1517	716	611	1627	2175
1733	11738	2409	957	754	1857	2564
1734	10752	2830	1228	829	1718	2212
1735	9672	1963	755	691	1605	2158
1736	10580	2706	993	816	2139	2445
1737	10054	2613	1008	885	2241	2652
1738	9600	2366	784	783	2072	2439
1739	9687	2302	844	875	1866	2218
1740	10765	2862	1235	947	2205	2783
1741	10456	2368	1072	1048	2816	3476
1742	9030	2642	1035	893	2203	2813
1743	8621	1955	947	813	1935	2342
1744	7394	1657	679	663	1744	2019
1745	7689	1631	672	626	1695	1940
1746	9503	2611	1089	895	2356	2728
1747	8741	2085	905	790	2190	2649
Totals of each Age.	194645	45984	19037	16575	40858	49709
$\frac{1}{20}$ Part for the mean Year.	9732	2299	952	829	2043	2485
1000 in the ſame Proportion	368	87	36	31	77	94

TABLE I.

ave died at the ſeveral Ages undermentioned; for
Mortality of the City of London.

40 & 50	50 & 60	60 & 70	70 & 80	80 & 90	90 & 100	100 & upwards.	Totals of each Year.
2624	2123	1863	1290	785	125	10	27810
2698	2338	1938	1375	769	137	6	29722
2373	1713	1577	1001	622	138	9	26761
2261	1839	1500	913	628	108	5	25262
2121	1741	1581	974	660	121	12	23358
2685	2196	1871	1188	804	198	12	29233
2154	1668	1324	793	484	66	4	26062
2138	1684	1339	872	565	84	12	23538
2357	2121	1666	1114	557	83	4	27581
2578	2270	1650	1164	576	127	5	27823
2363	2106	1551	1121	529	101	10	25825
2378	2039	1421	1166	547	76	13	25432
2866	2585	1977	1716	758	100	12	30811
3731	2851	1933	1540	734	130	14	32169
2959	2333	1634	1250	582	100	9	27483
2611	2004	1729	1507	629	93	14	25200
2123	1637	1307	920	387	73	3	20606
2332	1741	1382	1064	437	77	10	21296
2876	2243	1699	1444	625	78	10	28157
2717	2079	1544	1199	520	68	7	25494
50945	41311	32486	23611	12198	2083	181	529623
2547	2066	1625	1181	610	104	9	26481
96	78	61	44	23	4		

460. TABLE

TABLE II.

460. *Exhibiting the Probabilities of Life, from th Bills of Mortality of the City of* London.

Years.	Number of Perſons dying off.	Number of Perſons living.	Years.	Number of Perſons dying off.	Number of Perſons living.	Years.	Number of Perſons dying off.	Number of Perſons living.	Years.	Number of Perſons dying off.	Number of Perſons living.
0		1000	24	7	452	48	9	229	72	5	61
1	276	724	25	8	444	49	9	220	73	5	56
2	92	632	26	8	436	50	9	211	74	5	51
3	49	583	27	8	428	51	9	202	75	5	46
4	25	558	28	9	419	52	9	193	76	4	42
5	13	545	29	9	410	53	8	185	77	4	38
6	10	535	30	9	401	54	8	177	78	4	34
7	8	527	31	9	392	55	8	169	79	4	30
8	7	520	32	9	383	56	8	161	80	3	27
9	6	514	33	9	374	57	7	154	81	3	24
10	5	509	34	9	365	58	7	147	82	3	21
11	4	505	35	9	356	59	7	140	83	3	18
12	4	501	36	9	347	60	7	133	84	3	15
13	3	498	37	10	337	61	7	126	85	2	13
14	2	496	38	10	327	62	7	119	86	2	11
15	2	494	39	10	317	63	6	113	87	2	9
16	2	492	40	10	307	64	6	107	88	2	7
17	2	490	41	10	297	65	6	101	89	2	5
18	3	487	42	10	287	66	6	95	90	1	4
19	4	483	43	10	277	67	6	89	91	1	3
20	5	478	44	10	267	68	6	83	92	1	2
21	6	472	45	10	257	69	6	77	93	1	1
22	6	466	46	10	247	70	5	71	94	1	0
23	7	459	47	9	238	71	5	66	95	0	

TABLE III.

61. *Shewing the Probabilities of Life, from the Bills of Mortality, according to Mr.* SIMPSON.

Age.	N°. of Perſons.	Age.	N°. of Perſons.	Age.	N°. of Perſons.	Age.	N°. of Perſons.
born	1280 —410	20	462 —7	40	294 —10	60	130 —7
1	870 —170	21	455 —7	41	284 —10	61	123 —6
2	700 —65	22	448 —7	42	274 —10	62	117 —6
3	635 —35	23	441 —7	43	264 —9	63	111 —6
4	600 —20	24	434 —8	44	255 —9	64	105 —6
5	580 —16	25	426 —8	45	246 —9	65	99 —6
6	564 —13	26	418 —8	46	237 —9	66	93 —6
7	551 —10	27	410 —8	47	228 —8	67	87 —5
8	541 —9	28	402 —8	48	220 —8	68	81 —6
9	532 —8	29	394 —9	49	212 —3	69	75 —6
10	524 —7	30	385 —9	50	204 —8	70	69 —5
11	517 —7	31	376 —9	51	196 —8	71	64 —5
12	510 —6	32	367 —9	52	188 —8	72	59 —5
13	504 —6	33	358 —9	53	180 —8	73	54 —5
14	498 —6	34	349 —9	54	172 —7	74	49 —4
15	492 —6	35	340 —9	55	165 —7	75	45 —4
16	486 —6	36	331 —9	56	158 —7	76	41 —3
17	480 —6	37	322 —9	57	151 —7	77	38 —3
18	474 —6	38	313 —9	58	144 —7	78	35 —3
19	468 —6	39	304 —10	59	137 —7	79	32 —3
20	462	40	294	60	130	80	29

462. I have thought it proper to give both theſe Tables of the Probabilities of Life, but in the future Computations ſhall confine myſelf to that of Mr. *Simpſon*, becauſe his Tables of *Annuities on Lives*, will beſt ſuit my Purpoſe; but before any Thing on this Subject can be well underſtood, we muſt premiſe the following lemmatical Propoſitions.

463. *The Probability of the happening of any Event is in Proportion to the Chances which that Event has to happen to the Number of all the Chances which it has both to happen and to fail.* Thus ſuppoſe *one Die* was to be caſt, there is but *one Chance* that any one Number of Spots on it ſhall come up; and it is evident there are five Chances for it to fail, therefore the Probability that any particular Number ſhall come up, will be as 1 to 1 + 5, that is, as 1 to 6; and may be thus expreſſed, $\frac{1}{6}$. Thus if the *Octahedron*, or Body of eight Sides, had the eight Digits on them reſpectively, then the Probability of any one coming up on a Caſt, would be as $\frac{1}{8}$.

464. Therefore, if the *Die* and *Octahedron* were both caſt up together, the Probability that any two Numbers you ſhall name, come up together, will be as $\frac{1}{6} \times \frac{1}{8}$, or $\frac{1}{48}$. Again, if a *Dodecahedron*, or Body of 12 Sides, were added to the two former, with the proper Numbers on it; and all three were thrown together, the Probability that any three Numbers ſpecified, come up together, will be, as $\frac{1}{6} \times \frac{1}{8} \times \frac{1}{12}$, or $\frac{1}{576}$; and ſo of any other Number.

465. Hence, if a Sum of Money was to be expected on the coming up of any Number on the Die, 'tis plain the Value of ſuch an Expectation would be but a $\frac{1}{6}$ Part of that Sum; or if the receiving a Sum depends on the happening of any two Numbers on the *Die* and *Octahedron* together, the Value of the Expectation would be but $\frac{1}{48}$ Part of that Sum.

466. If, in the ſecond of the foregoing Tables, we find at the Age of 25 Years there are 444 People living out of 1000, and againſt the Age of 60, there are but 133, it appears that in that Interval there have died 311: So that the Number of Chances which a Perſon of 25 Years of Age has to live to 60, will be as 133, and the Chances he has to fail will be as 311; therefore his Probability of living to the Age of 60 Years will be as 133 to 133 + 321, or as $\frac{133}{444}$. (by 463.)

467. Le

467. Let $a =$ Number of Chances any Event has to happen, nd $b =$ the Number of all the Chances there are for it to appen and fail; and put $p =$ to the Probability of it hap-ening; then by what we have premiſed, we have this Theo-em, $p = \frac{a}{b}$, therefore $pb = a = a \times 1$; whence we have :p::b:a; and then $1 - p : p :: b - a : a$ (as will appear y multiplying Extremes and Means.) In the above Example, $= 133$, $b = 444$, and $p = \frac{a}{b} = 0,3$. The Chance there-ore a Perſon of 25 has to live to 60 is as 0,3 to 1, and not as 0,4276 to 1, as ſome have aſſerted.

468. If any Sum S be expected on the happening of any Event, the Value V of that Expectation will be in Proportion o the Sum depending, and the Probability of the Event's hap-ening, that is, V will be as p S; therefore V : p S : $\frac{aS}{b}$; and o we have b V $= a$ S; and V : S : : a : b.

469. Suppoſe a Perſon A was to receive 100 *l.* upon this Condition, that another, B, of 20 Years Age, ſhould live one Year; Query the Value of A's Expectation? In Table III. (for we ſhall uſe Mr. *Simpſon*'s Numbers for the future) againſt he Age of 20 and 21, the Numbers are 462, and 455; whence :b :: 455 : 462. Now the preſent Value S of 100 *l.* due at he End of one Year, allowing (ſuppoſe) 4 *l. per Cent.* is 96,15 *l.* (See Table III. p. 179.) Wherefore $V = \frac{aS}{b} = \frac{455 \times 96.15}{462}$ $= 94.7$ *l.* the true Value required.

470. In like Manner the Probability a Perſon of 20 Years of Age has of living 2 Years is $\frac{448}{462}$; and the preſent Worth of 00 *l.* due at the End of 2 Years (at 4 *per Cent.*) is 92,45; herefore $\frac{448}{462} \times 92,45 = 89,65$ *l.* the Value of A's Expecta-ion to receive 100 *l.* at the End of the 22d Year of B's Life. And thus you proceed for all the other Years of his Life to the Extremity of Age; and the Sum of all theſe being found, and dded together, will amount to 1480 *l.* very nearly. But if Table II. of Mr. *Stonehouſe* be uſed, then the Value of an An-uity for ſuch a Life will amount to 1485 *l.*

471. Since $\frac{A}{R^t} = P$ (402.) and putting $t = 1, 2, 3, 4,$ &c. and $A = 1$ *l.* 'tis plain the Sum of the Values of 1 *l.* fo the ſucceſſive Years of a Perſon's Life will be expreſſed by thi Series $\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \frac{1}{R^4}$, &c. were there no Contin-gency in the Caſe. But as we muſt allow for that; let Q b the Number in the Table, correſponding to the given Age of B and $\dot{Q}$. $\ddot{Q}$. $\dddot{Q}$. &c. be thoſe anſwering to the next ſucceedin Ages reſpectively; then we ſhall have $\frac{\dot{Q}}{QR} + \frac{\ddot{Q}}{QR^2} + \frac{\dddot{Q}}{QR}$ &c. or $\frac{1}{Q} \times \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2} + \frac{\dddot{Q}}{R^3}$, &c. equal to the Value of a Annuity of 1 *l.* on the Life of B.

472. When the Value (V) of any one Life is computed o given, the Value (*v*) of the next younger Life will be eaſil deduced from thence; for let $\dot{q}$ be the Number in the Tabl found againſt the next younger Age; then (by 471.) for th ſame Reaſon that $V = \frac{1}{Q} \times \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2} + \frac{\dddot{Q}}{R^3}$, &c. will $= \frac{Q}{R\dot{q}} + \frac{\dot{Q}}{R^2\dot{q}} + \frac{\ddot{Q}}{R^3\dot{q}}$, &c. $= \frac{1}{R\dot{q}} \times Q + \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2}$ &c. Whence 'tis plain, by multiplying the former Series by Q and the latter by $R\dot{q}$, we get $QV = \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2} + \frac{\dddot{Q}}{R^3}$, & and $vR\dot{q} = Q + \frac{\dot{Q}}{R} + \frac{\ddot{Q}}{R^2}$, &c. Therefore $vR\dot{q} -$ ($= QV$; and ſo $QV + Q = vR\dot{q}$; conſequently $v =$

$$\frac{QV + Q}{R\dot{q}} = \overline{V + 1} \times \frac{Q}{R\dot{q}} = \overline{V + 1} \times \frac{1}{R} \times \frac{Q}{\dot{q}}.$$

473. This laſt Theorem is thus expreſſed in Words; To th Value of the given Life (V) add one Year's Purchaſe, and mu tip

iply that Sum (V + 1) by $\frac{1}{R}$ (which is the ſame Thing as to iſcount it for one Year,) and that Product again multiply by he Probability of a Life of *Nineteen* continuing one Year (*viz.* $\frac{Q}{q}$); this laſt Product will be the Value (V) of an Annuity upon this Liſe. For Example, The Life of 20 being 1480 *l.* this encreaſed by one Year's Purchaſe is 1580 *l.* which diſcounted at 4 *per Cent.* is 1519.2; this multiplied by the Probability of a Life of Nineteen, *viz.* $\frac{462}{468}$ gives 1499,8, or 1500 *l.* for the required Value of ſuch a Life.

474. If the Annuity was but 1 *l.* inſtead of 100 *l.* the Value of ſuch an Annuity on a Life of 20, would have been 14,8 *l.* and for a Life of 19, it would be 15 *l.* and in ſuch a Manner are the Numbers in the following Table IV. computed, which ſhews the Value of an Annuity on any ſingle Life from ſix Years of Age to 75 (according to Mr. *Simpſon*,) at the Rate of 3, 4, 5, *per Cent.*

N. B. Compound Intereſt is always allowed in theſe Caſes. And the Value of an Annuity on any Life, is called ſuch a Number of *Years Purchaſe*, as it is the preſent Value of the Annuity to continue for ſo many Years certain. Thus the preſent Value of an Annuity of 100 *l.* *per Annum* on a Life of 20 Years of Age, diſcounting 4 *per Cent.* is equal to $14,8 \times 100 =$ 1480 *l.* as per Calculation (470.)

TABLE

TABLE II.

For the Valuation of Annuities upon one LIFE.

Age.	Year's Purch. at 5 *per Cent.*	Year's Purch. at 4 *per Cent.*	Year's Purch. at 3 *per Cent*	Age.	Year's Purch. at 5 *per Cent.*	Year's Purch. at 4 *per Cent.*	Year's Purch. at 3 *per Cent.*	Age.	Year's Purch. at 5 *per Cent.*	Year's Purch. at 4 *per Cent.*	Year's Purch. at 3 *per Cent*
6	14.1	16.2	18.8	31	11.4	12.9	14.8	56	8.4	9.1	10.1
7	14.2	16.3	18.9	32	11.3	12.7	14.6	57	8.2	8.9	9.9
8	14.3	16.4	19.0	33	11.2	12.6	14.4	58	8.1	8.7	9.6
9	14.3	16.4	19.0	34	11.0	12.4	14.2	59	8.0	8.6	9.4
10	14.3	16.4	19.0	35	10.9	12.3	14.1	60	7.9	8.4	9.2
11	14.3	16.4	19.0	36	10.8	12.1	13.9	61	7.7	8.2	8.9
12	14.2	16.3	18.9	37	10.6	11.9	13.7	62	7.6	8.1	8.7
13	14.1	16.2	18.7	38	10.5	11.8	13.5	63	7.4	7.9	8.5
14	14.0	16.0	18.5	39	10.4	11.6	13.3	64	7.3	7.7	8.3
15	13.9	15.8	18.3	40	10.3	11.5	13.2	65	7.1	7.5	8.0
16	13.7	15.6	18.1	41	10.2	11.4	13.0	66	6.9	7.3	7.8
17	13.5	15.4	17.9	42	10.1	11.2	12.8	67	6.7	7.1	7.6
18	13.4	15.2	17.6	43	10.0	11.1	12.6	68	6.6	6.9	7.4
19	13.2	15.0	17.4	44	9.9	11.0	12.5	69	6.4	6.7	7.1
20	13.0	14.8	17.2	45	9.8	10.8	12.3	70	6.2	6.5	6.9
21	12.9	14.7	17.0	46	9.7	10.7	12.1	71	6.0	6.3	6.7
22	12.7	14.5	16.8	47	9.5	10.5	11.9	72	5.8	6.1	6.5
23	12.6	14.3	16.5	48	9.4	10.4	11.8	73	5.6	5.9	6.2
24	12.4	14.1	16.3	49	9.3	10.2	11.6	74	5.4	5.6	5.9
25	12.3	14.0	16.1	50	9.2	10.1	11.4	75	5.2	5.4	5.6
26	12.1	13.8	15.9	51	9.0	9.9	11.2				
27	12.0	13.6	15.6	52	8.9	9.8	11.0				
28	11.8	13.4	15.4	53	8.8	9.6	10.7				
29	11.7	13.2	15.2	54	8.6	9.4	10.5				
30	11.6	13.1	15.0	55	8.5	9.3	10.3				

476. We might now proceed to compute the Value of An-uities on two, three, or more *joint* Lives, or on the longeſt Liver f them, on the ſame Principles as before; but we ſhall deſiſt r the preſent, from any further Inveſtigations of this Kind, r two Reaſons; firſt, becauſe of the great Difficulty and In-icacy of them; and, ſecond, becauſe they are not ſo neceſ-ary, or ſo often in Uſe as Annuities on a ſingle Life. And to y, what I think is Truth, the whole Affair of *buying and ſell-g Annuities* on ſeveral Lives, ſeems to be a Sort of *gaming*, at aſt a Matter of great Uncertainty after all; for general Rules educed from the beſt Bills of Mortality that could be had, muſt e very fallacious, and ſhort of aſcertaining the preſent real Va-ue of an Annuity on any particular Life or Lives, and can be ooked upon only as affording *Mediums* and *Approximations* hereto. As we have ſhewn the general Nature of this *Calculus*, nd given the moſt uſeful Part, we ſhall next proceed to hat remains of the *Algebraic Inſtitutions*.

CHAP. XXII.

Of the Nature, Geneſis, and Roots of Cubic Equations, and thoſe of higher Dimenſions.

77. AS we have been ſufficiently prolix on the Nature and Geneſis of *Quadratic Equations* (in Chap. XIV.) the eſs will be neceſſary to be ſaid here in regard to *Cubic Ones*, and hoſe of higher Powers. For the Nature and Rationale of all epend on that of the *component, or generating Roots*. Thus, uppoſe the Value of the unknown Quantity x in any Equation ere to be expreſſed by a, b, c, d, &c. that is, let $x = a$, $x = b$, $x = c$, $x = d$, &c. then will $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$, &c. be the ſimple *radical Equations*, f which thoſe of higher Orders are compoſed; and as the Pro-uct of any two of theſe gives a *Quadratic Equation*, or one of wo Dimenſions; ſo the Product of any three of them as $\overline{x - a} \times \overline{x - b} \times \overline{x - c} = 0$, will give a *Cubic Equation* or one of

three Dimenſions. And the Product of four of them will co ſtitute a *Biquadratic Equation* or one of four Dimenſions; and on. Therefore, in general, *the higheſt Dimenſion of the unkn Quantity x is equal to the Number of ſimple Equations that are m tiplied together to produce it.*

478. When any Equation equivalent to this Biquadra $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$ is propoſed to be r ſolved, the whole Difficulty conſiſts in finding the ſimple Equ tions $\overline{x-a}=0$, $\overline{x-b}=0$, $\overline{x-c}=0$, $\overline{x-d}=0$, b whoſe Multiplication it is produced; for each of theſe ſimp Equations gives one of the Values of x, and one Solution of th propoſed Equation. For, if any of the Values of x deduc from thoſe ſimple Equations be ſubſtituted in the propoſed Equ tion, in place of x, then all the Terms of that Equation will v niſh, and the whole be found equal to nothing. Becauſe whe it is ſuppoſed that $x=a$, or $x=b$, or $x=c$, or $x=d$, th the Product $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}$ does vaniſ becauſe one of the Factors is equal to nothing. There are ther fore four Suppoſitions that give $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-}$ $=0$ according to the propoſed Equation; that is, there are fo Roots of the propoſed Equation. And after the ſame Mann *any other* Equation admits of as many Solutions as there are ſi ple Equations multiplied by one another that produce it, or many as there are Units in the higheſt Dimenſion of the u known Quantity in the propoſed Equation.

479. But as there are no other Quantities whatſoever beſid theſe four $(a, b, c, d,)$ that ſubſtituted in the Product $x-$ $\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}$, in the Place of x, will make t Product vaniſh; therefore, the Equation $\overline{x-a}\times\overline{x-b}$ $\overline{x-c}\times\overline{x-d}=0$, cannot poſſibly have more than theſe fo Roots, and cannot admit of more Solutions than four. If y ſubſtitute in that Product a Quantity neither equal to a, n b, nor c, nor d, which ſuppoſe e, then ſince neither, $e-$ $e-b$, $e-c$, nor $e-d$ is equal to nothing; their Product $e-$ $\times\overline{e-b}\times\overline{e-c}\times\overline{e-d}$ cannot be equal to nothing, b muſt be ſome real Product: And therefore, there is no Su

poſitio

oſition beſide one of the foreſaid Four, that gives a juſt Va-
ue of x according to the propoſed Equation. So that it can
ave no more than theſe four Roots. And after the ſame Man-
er it appears, that *no Equation can have more Roots than it con-
ains Dimenſions of the unknown Quantity.*

480. To make all this ſtill plainer by an Example, in Num-
ers; ſuppoſe the Equation to be reſolved to be $x^4 - 10x^3 +$
$35x^2 - 50x + 24 = 0$, and that you diſcover that this Equa-
ion is the ſame with the Product of $\overline{x-1} \times \overline{x-2} \times \overline{x-3}$
$\times \overline{x-4}$, then you certainly infer that the four Values of x
re 1, 2, 3, 4; ſeeing any of theſe Numbers, placed for x,
makes that Product, and conſequently $x^4 - 10x^3 + 35x^2 -$
$50x + 24$, equal to nothing, according to the propoſed Equa-
ion. And it is certain that there can be no other Values of x
eſides theſe four: Since, when you ſubſtitute any other Num-
er for x in thoſe Factors $x-1$, $x-2$, $x-3$, $x-4$, none
f the Factors vaniſh, and therefore their Product cannot be
qual to nothing, according to the Equation.

481. *The Number of Terms is always greater than the higheſt Di-
enſion of the unknown Quantity by Unit.* And when any Term
s wanting, an *Aſteriſk* is marked in its Place. The *Signs* and
Coefficients of Equations will be underſtood by conſidering the fol-
owing Table, where the ſimple Equations $x-a$, $x-b$, *&c.*
re multiplied by one another, and produce, ſucceſſively, the
igher Equations.

$$x - a = 0$$

$$\times\; \underline{x - b = 0}$$

$$= \left.\begin{array}{l} x^2 - a\,x \\ \quad\;\; - b\,x + a\,b \end{array}\right\} = 0, \text{ a } \textit{Quadratic.}$$

$$\times\; \underline{x - c = 0}$$

$$= x^3 \left.\begin{array}{r} -a \\ -b \\ -c \end{array}\right\} \times x^2 \left.\begin{array}{r} +ab \\ +ac \\ +bc \end{array}\right\} \times x - abc = 0, \text{ a } \textit{Cubic.}$$

$$\times\; \underline{x - d = 0}$$

$$= x^4 \left.\begin{array}{r} -a \\ -b \\ -c \\ -d \end{array}\right\} \times x^3 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} \times x^2 \left.\begin{array}{r} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} \times x + abcd = 0, \;\; \textit{Biquadratic.}$$

$$\times\; \overline{x - e} = 0$$

$$= x^5 \left.\begin{array}{r} -a \\ -b \\ -c \\ -d \\ -e \end{array}\right\} \times x^4 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +ae \\ +bc \\ +bd \\ +be \\ +cd \\ +ce \\ +de \end{array}\right\} \times x^3 \left.\begin{array}{r} -abc \\ -abd \\ -abe \\ -acd \\ -ade \\ -ace \\ -bcd \\ -bce \\ -bde \\ -cde \end{array}\right\} \times x^2 \left.\begin{array}{r} +abcd \\ +abce \\ +abde \\ +acde \\ +bcde \end{array}\right\} \times x - abcde = 0.$$

(a *Surſolid.*)

&c.

482. From the Inſpection of theſe Equations it is plain, that the Coefficients of the firſt Term is *Unit.*

The Coefficient of the ſecond Term is *the Sum of all the Roots* (a, b, c, d, e,) *having their Signs changed.*

The Coefficient of the third Term is *the Sum of all the Product, that can be made by multiplying any two of the Roots* (a, b, c, d, e,) *by one another.*

The Coefficient of the fourth Term is *the Sum of all the Products that can be made by multiplying into one another any three of the Roots, with their Signs changed.* And after the ſame Manner all the other Coefficients are formed.

The laſt Term is always *the Product of all the Roots having their Signs changed,* multiplied by one another.

483, Altho' in the Table ſuch ſimple Equations only are multiplied by one another as have poſitive Roots, it is eaſy to ſee, "that the Coefficients will be formed according to the ſame Rule when any of the ſimple Equations have negative Roots." And, *in general*, if $x^3 - px^2 + qx - r = 0$ repreſent any Cubic Equation, then ſhall p be the Sum of the Roots; q the Sum of the Products made by multiplying any two of them; r the Product of all the three: And, if $-p$, $+q$, $-r$, $+s$, $-t$, $+u$, &c. be the Coefficients of the 2d, 3d, 4th, 5th, 6th, 7th, *&c.* Terms of any Equation, then ſhall p be the Sum of all the Roots, q the Sum of the Products of any two, r the Sum of the Products of any Three, s the Sum of the Products of any Four, t the Sum of the Products of any Five, u the Sum of the Products of any Six, *&c.*

484. When therefore any Equation is propoſed to be reſolved, it is eaſy to find the Sum of the Roots, (for it is equal to the Coefficient of the ſecond Term having its Sign changed:) or, to find the Sum of the Products that can be made by multiplying any determinate Number of them.

485. But it is alſo eaſy "to find the Sum of the Squares, or of any Powers, of the Roots."

The Sum of the Squares is always $p^2 - 2q$. For calling the Sum of the Squares B, ſince the Sum of the Roots is p; and "the Square of the Sum of any Quantities is always equal to the Sum of their Squares added to double the Products that can be made by multiplying any two of them," therefore, $p^2 = B + 2q$, and conſequently, $B = p^2 - 2q$. For Example, $\overline{a + b + c}|^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. That is $p^2 = B + 2q$. And $\overline{a + b + c + d}|^2 = a^2 + b^2 + c^2 + d^2 + 2 \times \overline{ab + ac + ad + bc + bd + cd}$, that is again $p^2 = B + 2q$, or $B = p^2 - 2q$. And ſo for any other Number of Quantities. In general therefore, "B the Sum of the Squares of the Roots may always be found by ſubtracting $2q$ from p^2;" the Quantities p and q being always known, ſince they are the Coefficients in the propoſed Equation.

486. "The Sum of the Cubes of the Roots of any Equation is equal to $p^3 - 3pq + 3r$, or to $Bp - pq + 3r$." For $B - q \times p$ gives always the Exceſs of the Sum of the Cubes of any Quantities above the triple Sum of the Products

that can be made by multiplying any three of them. Thu $\overline{a^2+b^2+c^2-ab-ac-bc}\times\overline{a+b+c}$ $(=\overline{B-q}\times$ $p) = a^3+b^3+c^3-3abc$. Therefore, if the Sum of th Cubes is called C, then ſhall $\overline{B-q}\times p = C-3r$, and $= Bp-qp+3r =$ (becauſe $B = p^2-2q$) $= p^3-3p$ $+\,3r$.

After the ſame Manner, if D be the Sum of the 4th Power of the Roots, you will find that $D = pC-qB+pr-4$ and if E be the Sum of the 5th Powers, then ſhall $E = pD-$ $C+rB-ps+5t$. And after the ſame Manner the Sum of any Powers of the Roots may be found; the Progreſſion of theſe Expreſſions of the Sum of the Powers being obvious.

487. As for the Signs of the Terms of the Equation produced, it appears from Inſpection that the Signs of all the Terms in any Equation in the Table are alternately + and —: Theſe Equations are generated by multiplying continually $x-a$, $x-b$, $x-c$, $x-d$, &c. by one another. The firſt Term is always ſome Pure Power of x, and is poſitive; the ſecond is a Power of x multiplied by the Quantities $-a$, $-b$, $-c$, &c. And ſince theſe are all negative, that Term muſt therefore be negative. The third Term has the Products of any two of theſe Quantities ($-a$, $-b$, $-c$, &c.) fo its Coefficient; which Products are all poſitive, becauſe $-\times-$ gives +. For the like Reaſon, the next Coefficient, conſiſting of all the Products made by multiplying any three of theſe Quantities, muſt be negative: And the next poſitive. So that the Coefficients in this Caſe, will be poſitive and negative by Turns. Bu "in this Caſe, the Roots are all poſitive;" ſince $x = a$, $x =$ $x = c$, $x = d$, $x = e$, &c. are the aſſumed ſimple Equations. It is plain then, that, "*when all the Roots are poſitive, the Sig are alternately + and —.*"

488. But if the Roots are all negative, then $\overline{x+a}\times\overline{x+}$ $\times\,\overline{x+c}\times\overline{x+d}$, &c. $= 0$ will expreſs the Equation to be produced; all whoſe Terms will plainly be poſitive; ſo that "*when all the Roots of an Equation are negative, it is plain there will be no changes in the Signs of the Terms of that Equation.*"

489. In general, "there are as many poſitive Roots in any Equation as there are Changes in the Signs of the Terms from

+ to —, or from — to +; and the remaining Roots are egative." The Rule is general, if the impoſſible Roots be llowed to be either poſitive or negative.

490. In Quadratic Equations, the two Roots are either both oſitive, as in this

$$(\overline{x-a}\times\overline{x-b}=)\ x^2 \begin{array}{l}-ax\\-bx\end{array}+ab=0,$$

here there are two Changes of the Signs: Or, they are both egative as in this

$$(\overline{x+a}\times\overline{x+b}=)\ x^2 \left.\begin{array}{l}+a\\+b\end{array}\right\}x+ab=0,$$

here there is not any Change of the Signs. Or there is one oſitive and one negative, as in

$$(\overline{x-a}\times\overline{x+b}=)\ x^2 \left.\begin{array}{l}-a\\+b\end{array}\right\}x-ab=0,$$

here there is neceſſarily one Change of the Signs; becauſe the irſt Term is poſitive, and the laſt negative, and there can be ut one Change whether the 2d Term be + or —.

Therefore the Rule given (Inſt. 489.) extends to all Quadratic Equations.

491. In Cubic Equations, the Roots may be,

1°. All-poſitive as in this, $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$, n which the Signs are alternately + and —, as appears from he Table; and there are three Changes of the Signs.

2°. The Roots may be all negative as in the Equation $\overline{x+a}\times\overline{x+b}\times\overline{x+c}=0$, where there can be no Change f the Signs. Or,

3°. There may be two poſitive Roots and one negative, as n the Equation $\overline{x-a}\times\overline{x-b}\times\overline{x+c}=0$; which gives

$$x^3 \left.\begin{array}{l}-a\\-b\\+c\end{array}\right\}x^2 \left.\begin{array}{l}+ab\\-ac\\-bc\end{array}\right\}x+abc=0.$$

Here there muſt be two Changes of the Signs, becauſe if $a +$ b is greater than c, the ſecond Term muſt be negative, its Coefficient being $-a-b+c$.

And if $a+b$ is leſs than c, then the third Term muſt be negative, its Coefficient $+ab-ac-bc$ $(ab-c\times\overline{a+b})$ being in that Caſe negative. And there cannot poſibly be thre Changes of the Signs, the firſt and laſt Terms having the ſam Sign.

492. 4°. There may be one poſitive Root and two negativ as in the Equation $\overline{x+a}\times\overline{x+b}\times\overline{x-c}=0$, which gives

$$x^3\left.\begin{array}{l}+a\\+b\\-c\end{array}\right\}x^2\left.\begin{array}{l}+ab\\-ac\\-bc\end{array}\right\}x-abc=0.$$

Where there muſt be always one Change of the Signs, ſince th firſt Term is poſitive and the laſt negative. And, there can b but one Change of the Signs, ſince if the ſecond Term i negative, or $a+b$ leſs than c, the third muſt be negativ alſo, ſo that there will be but one Change of the Signs. Or if the ſecond Term is affirmative, whatever the third Term is there will be but one Change of the Signs. It appears therefore in general, that in Cubic Equations, there are as many affirmative Roots as there are Changes of the Signs of the Terms of th Equation.

The ſame Way of Reaſoning may be extended to Equations of higher Dimenſions, and the Rule delivered in Inſt. 489. extended to all Kinds of Equations.

493. There are ſeveral Conſectaries of what has been already demonſtrated, that are of Uſe in diſcovering the Roots o Equations. But before we proceed to that, it will be convenient to explain ſome Transformations of Equations, by which they may often be rendered more ſimple, and the Inveſtigation of their Roots more eaſy.

CHAP

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* Becauſe the Rectangle $a\times b$ is leſs than the Square $\overline{a+b}\times\overline{a+b}$ and therefore much leſs than $\overline{a+b}\times c$.

CHAP. XXIII.

f the *Transformation of Equations; and exterminating their intermediate Terms.*

94. WE now proceed to explain the Transformations of Equations that are moſt uſeful: And firſt, *The Af- rmative Roots of an Equation are changed into negative Roots of the me Value, and the negative Roots into affirmative, by only changing e Signs of the Terms alternately, beginning with the Second.* Thus, e Roots of the Equation $x^4 - x^3 - 19x^2 + 49x - 30 =$ are $+1$, $+2$, $+3$, -5; whereas the Roots of the ſame quation having only the Signs of the ſecond and fourth Terms hanged, *viz.* $x^4 + x^3 - 19x^2 - 49x^2 - 30 = 0$ are -1, -2, -3, $+5$.

To underſtand the Reaſon of this Rule, let us aſſume an Equaon, as $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e}$, &c. $= 0$, hoſe Roots are $+a$, $+b$, $+c$, $+d$, $+e$, &c. and ancher having its Roots of the ſame Value, but affected with ntrary Signs, as $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e}$, c. $= 0$. It is plain, that the Terms taken alternately, beinning from the firſt,* are the ſame in both Equations, and have e ſame Sign, “being Products of an even Number of the oots;” the Product of any two Roots having the ſame Sign as eir Product when both their Signs are changed; as $+a \times -$ $= -a \times +b$.

But the ſecond Terms, and all taken alternately from them, ecauſe their Coefficients involve always the Products of an dd Number of the Roots, will have contrary Signs in the two quations. For Example, the Product of four *viz.* $abcd$, aving the ſame Sign in both, and one Equation in the fifth erm having $abcd \times +e$, and the other $abcd \times -e$, it llows, that their Product $abcde$ muſt have contrary Signs in e two Equations: Theſe two Equations therefore that have e ſame Roots, but with contrary Signs, have nothing different ut the Signs of the alternate Terms, beginning with the ſeond. From which it follows, “that if any Equation is given and you change the Signs of the alternate Terms, beginn with the ſecond, the new Equation will have Roots of the ſa Value, but with contrary Signs."

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* See the Table in Page 238.

495. It is often very uſeful "*to transform an Equation another that ſhall have its Roots greater or leſs than the Root the propoſed Equation by ſome given Difference.*"

Let the Equation propoſed be the Cubic $x^3 - px^2 + qx$ $r = 0$. And let it be required to transform it into another Eq tion whoſe Roots ſhall be leſs than the Roots of this Equat by ſome given Difference (e), that is, ſuppoſe $y = x - e$, conſequently $x = y + e$; then inſtead of x and its Pow ſubſtitute $y + e$ and its Powers, and there will ariſe this Equation

$$(A)\quad \left.\begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ \quad\;\; - py^2 - 2pey - pe^2 \\ \qquad\qquad\;\; + qy + qe \\ \qquad\qquad\qquad\quad - r \end{array}\right\} = 0$$

whoſe Roots are leſs than the Roots of the preceding Eq tion by the Difference (e).

If it had been required to find an Equation whoſe Roots ſho be greater than thoſe of the propoſed Equation by the Qu tity (e), then we muſt have ſuppoſed $y = x + e$, and co quently $x = y - e$, and then the other Equation would h had this Form

$$(B)\quad \left.\begin{array}{l} y^3 - 3ey^2 + 3e^2y - e^3 \\ \quad\;\; - py^2 + 2pey - pe^2 \\ \qquad\qquad\;\; + qy - qe \\ \qquad\qquad\qquad\quad - r \end{array}\right\} = 0$$

If the propoſed Equation be in this Form $x^3 + px^2 +$ $+ r = 0$, then by ſuppoſing $x + e = y$ there will ariſe an quation agreeing in all Reſpects with the Equation (A), that the ſecond and fourth Terms will have contrary Signs.

And by ſuppoſing $x - e = y$, there will ariſe an Equa agreeing with (B) in all Reſpects, but that the ſecond fourth Terms will have contrary Signs to what they h in (B).

The firſt of theſe Suppoſitions gives this Equation,

$$(C)\ \left.\begin{array}{l} y^3 - 3ey^2 + 3e^2y - e^3 \\ \quad + py^2 - 2pey + pe^2 \\ \qquad + qy - qe \\ \qquad\quad + r \end{array}\right\} = 0$$

The ſecond Suppoſition gives the Equation,

$$(D)\ \left.\begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ \quad + py^2 + 2epy + pe^2 \\ \qquad + qy + qe \\ \qquad\quad + r \end{array}\right\} = 0$$

496. The firſt Uſe of this Transformation of Equations is to *w how the ſecond (or other intermediate) Term may be taken away of an Equation.*

It is plain that in the Equation *(A)* whoſe ſecond Term is $\overline{-p} \times y^2$, if you ſuppoſe $e = \frac{1}{3}p$, and conſequently $3e -$ $= 0$, then the ſecond Term will vaniſh.

In the Equation *(C)* whoſe ſecond Term is $-\overline{3e + p} \times y^2$, poſing $e = \frac{1}{3}p$, the ſecond Term alſo vaniſhes.

Now the Equation *(A)* was deduced from $x^3 - px^2 + qx$ $r = 0$, by ſuppoſing $y = x - e$: And the Equation *(C)* s deduced from $x^3 + px^2 + qx + r = 0$; by ſuppoſing $= x + e$. From which this Rule may eaſily be deduced for erminating the ſecond Term out of any Cubic Equation.

RULE.

497. *Add to the unknown Quantity of the given Equation the d Part of the Coefficient of the ſecond Term with its proper Sign,* $\mp \frac{1}{3}p$, *and ſuppoſe this Aggregate equal to a new unknown antity* (y). *From this Value of* y *find a Value of* x *by Tranſpo- n, and ſubſtitute this Value of* x *and its Powers in the given uation, and there will ariſe a new Equation that ſhall want the d Term.*

EXAMPLE.

Let it be required to exterminate the ſecond Term out of Equation, $x^3 - 9x^2 + 26x - 34 = 0$, ſuppoſe $x - 3$ y, or $y + 3 = x$; and ſubſtituting according to the Rule, will find

$$\left.\begin{array}{rrrr} y^3 & +\,9y^2 & +\,27y & +\,27 \\ & -\,9y^2 & -\,54y & -\,81 \\ & & +\,26y & +\,78 \\ & & & -\,34 \end{array}\right\} = 0$$

$$y^3 \quad * \quad -y \quad -10 = 0.$$

In which there is no Term where y is of two Dimenſions, an an Aſteriſk is placed in the room of the ſecond Term, to ſhe it is wanting.

498. Let the Equation propoſed be of any Number of Di menſions repreſented by (n); and let the Coefficient of th ſecond Term with its Sign prefixed be $-p$, then ſuppoſing x $\frac{p}{n} = y$, and conſequently $x = y + \frac{p}{n}$, and ſubſtituting this Va lue for x in the given Equation, there will ariſe a new Equatio that ſhall want the ſecond Term.

It is plain from what was demonſtrated in *Chap* 2. that th Sum of the Roots of the propoſed Equation is $+ p$, and ſinc we ſuppoſe $y = x - \frac{p}{n}$, it follows, that, in the new Equatio each Value of y will be leſs than the reſpective Value of x b $\frac{p}{n}$; and, ſince the Number of the Roots is n, it follows th the Sum of the Values of y will be leſs than $+ p$, the Sum the Values of x, by $n \times \frac{p}{n}$, that is, by $+ p$: Therefore th Sum of the Values of y will be $+ p - p = 0$.

But the Coefficient of the ſecond Term of the Equation y is the Sum of the Values of y, *viz.* $+ p - p$, and therefo that Coefficient is equal to nothing; and conſequently, in th Equation of y, the ſecond Term vaniſhes. It follows the that the ſecond Term may be exterminated out of any give Equation by the following

RULE.

Divide the Coefficient of the ſecond Term of the propoſed Equati by the Number of Dimenſions of the Equation; and aſſuming a ne unknown Quantity y, *add to it the Quotient having its Sign change*

Th

hen ſuppoſe this Aggregate equal to x the unknown Quantity in the ropoſed Equation; and for x and its Powers, ſubſtitute the Aggre- te and its Powers, ſo ſhall the new Equation that ariſes want its ond Term.

499. If the propoſed Equation is a Quadratic, as $x^2 - px +$ $= 0$, then, according to the Rule, ſuppoſe $y + \frac{1}{2}p = x$, d ſubſtituting this Value for x, you will find,

$$\left.\begin{array}{r} y^2 + py + \frac{1}{2}p^2 \\ - py - \frac{1}{2}p^2 \\ + \quad q \end{array}\right\} = 0$$

$$y^2 \quad * \quad - \frac{1}{4}p^2 + q = 0.$$

nd from this Example the Uſe of exterminating the 2d Term pears: For commonly the Solution of the Equation that wants e 2d Term is more eaſy. And, if you can find the Value of from this new Equation, it is eaſy to find the Value of x by eans of the Equation $y + \frac{1}{2}p = x$. For Example,

Since $y^2 + q - \frac{1}{4}p^2 = 0$, it follows that

$y^2 = \frac{1}{4}p^2 - q$, and $y = \pm\sqrt{\frac{1}{4}p^2 - q}$, ſo

that $x = y + \frac{1}{2}p = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$;

ich agrees with what we demonſtrated (Inſt. 339.)

If the propoſed Equation is a Biquadratic, as $x^4 - px^3 +$ $^2 - rx + s = 0$, then by ſuppoſing $x - \frac{1}{4}p = y$ or $x = y$ $\frac{1}{4}p$, an Equation ſhall ariſe having no ſecond Term. And if e propoſed is of five Dimenſions, then you muſt ſuppoſe $x = y$ $\frac{1}{5}p$. And ſo on.

500. When the ſecond Term in any Equation is wanting, it llows, that "the Equation has both affirmative and negative ots," and that the Sum of the affirmative Roots is equal to the m of the negative Roots." By which Means the Coefficient the ſecond Term, which is the Sum of all the Roots of both rts, vaniſhes, and makes the ſecond Term vaniſh.

In general, "the Coefficient of the ſecond Term is the Dif- ence between the Sum of the affirmative Roots and the Sum the negative Roots:" And the Operations we have given ve only to diminiſh all the Roots when the Sum of the Af-

firmative

firmative is greateſt, or increaſe the Roots when the Sum o the Negative is greateſt, ſo as to balance them, and reduce the to an Equality.

It is obvious, that in a Quadratic Equation that wants th ſecond Term, there muſt be one Root affirmative and one ne gative ; and theſe muſt be equal to one another.

In a Cubic Equation that wants the ſecond Term, there mu be either two affirmative Roots equal, taken together, to a thir Root that muſt be negative; or, two negative equal to a thir that muſt be poſitive.

501. Let an Equation $x^3 - px^2 + qx - r = 0$ be propoſe and let it be now required to exterminate the third Term.

By ſuppoſing $y = x - e$, the Coefficient of the third Ter in the Equation of y is found (ſee Equation *A*) to be $3e^2 -$ $2pe + q$. Suppoſe that Coefficient equal to nothing, and b reſolving the Quadratic Equation $3e^2 - 2pe + q = 0$, yo will find the Value of e, which ſubſtituted for it in the Equatio $y = x - e$, will ſhew how to transform the propoſed Equa tion into one that ſhall want the third Term.

The Quadratic $3e^2 - 2pe + q = 0$ gives $e = \frac{p \pm \sqrt{p^2 - 3}}{3}$ So that the propoſed Cubic will be transformed into an Equatio wanting the third Term by ſuppoſing $y = x - \frac{p - \sqrt{p^2 - 3}}{3}$ or $y = x - \frac{p + \sqrt{p^2 - 3q}}{3}$.

502. If the propoſed Equation is of n Dimenſions, the Value e by which the third Term may be taken away, is had by r ſolving the Quadratic Equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times \overline{n-1}}$ 0, ſuppoſing $-p$ and $+q$ to be the Coefficients of the ſecon and third Terms of the propoſed Equation.

The 4th Term of any Equation may be taken away by ſol ing a Cubic Equation, which is the Coefficient of the 4th Ter in the Equation when transformed, as in the ſecond Article this Chapter. The 5th Term may be taken away by ſolving Biquadratic; and after the ſame Manner the other Terms can exterminated if there are any.

503. The

503. There are other Tranſmutations of Equations that on ſome Occaſions are uſeful.

An Equation as $x^3 - px^2 + qx - r = 0$, *may be tranſformed into another that ſhall have its Roots equal to the Roots of this Equation multiplied by a given Quantity*, as f, by ſuppoſing $y = fx$, and conſequently $x = \frac{y}{f}$, and ſubſtituting this Value for x in the propoſed Equation, there will ariſe $\frac{y^3}{f^3} - \frac{py^2}{f^2} + \frac{qy}{f} - r = 0$, and multiplying all by f^3, we have $y^3 - fpy^2 + f^2 qy - f^3 r = 0$, where the Coefficient of the ſecond Term of the propoſed Equation multiplied into f makes the Coefficient of the ſecond Term of the transformed Equation; and the following Coefficients are produced by the following Coefficients of the propoſed Equation (as q, r, &c.) multiplied into the Powers of f, (as f^2, f^3, &c.)

Therefore "to transform any Equation into another whoſe Roots ſhall be equal to the Roots of the propoſed Equation multiplied by a given Quantity" (f), you need only multiply the Terms of the propoſed Equation, beginning at the ſecond Term, by f, f^2, f^3, f^4, &c. and putting y inſtead of x, there will ariſe an Equation having its Roots equal to the Roots of the propoſed Equation, multiplied by (f) as required.

504. The Transformation mentioned in the laſt Article is of Uſe when the higheſt Term of the Equation has a Coefficient different from Unity; for, by it, the Equation may be transformed into one that ſhall have the Coefficient of the higheſt Term Unit.

If the Equation propoſed is $ax^3 - px^2 + qx - r = 0$, then transform the Equation into one whoſe Roots are equal to the Roots of the propoſed Equation multiplied by (a). That is, ſuppoſe $y = ax$ or $x = \frac{y}{a}$ and there will ariſe

$$\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0; \text{ ſo that}$$

$$y^3 - py^2 + qay - ra^2 = 0.$$

From which we eaſily draw this

RULE

RULE.

Change the unknown Quantity x *into another* y, *prefix no Coefficient to the higheſt Term, paſs the ſecond, multiplying the following Terms, beginning with the third, by* a, a^2, a^3, a^4, *&c. the Powers of the Coefficient of the higheſt Term of the propoſed Equation, reſpectively.*

Thus the Equation $3x^3 - 13x^2 + 14x + 16 = 0$, is transformed into the Equation $y^3 - 13y^2 + 14 \times 3 \times x + 16 \times 9 = 0$, or $y^3 - 13y^2 + 42x + 144 = 0$.

Then finding the Roots of this Equation it will eaſily be diſcovered what are the Roots of the propoſed Equation: Since $3x = y$, or $x = \frac{1}{3}y$. And therefore ſince one of the Values of y is -2, it follows that one of the Values of x is $-\frac{2}{3}$.

505. By the laſt Rule *an Equation is eaſily cleared of Fractions.*

Suppoſe the Equation propoſed is $x^3 - \frac{p}{m}x^2 + \frac{q}{n}x - \frac{r}{e} = 0$. Multiplying all the Terms by the Product of the Denominators, you find

$$mne \times x^3 - nep \times x^2 + meq \times x - mnr = 0.$$

Then, (by laſt Art.) transforming the Equation into one that ſhall have Unit for the Coefficient of the higheſt Term, you find

$$y^3 - nep \times y^2 + m^2 e^2 nq \times y - m^3 n^3 e^2 r = 0.$$

Or, neglecting the Denominator of the laſt Term $\frac{r}{e}$, you need only multiply all the Equation by mn, which will give

$$mn \times x^3 - np \times x^2 + mq \times x - \frac{mnr}{e} = 0.$$ And

then $$y^3 - np \times y^2 + m^2 nq \times y - \frac{m^3 n^3 r}{e} = 0.$$

Now after the Values of y are found, it will be eaſy to diſcover the Values of x; ſince, in the firſt Caſe, $x = \frac{y}{mne}$; in the ſecond, $x = \frac{y}{mn}$.

For Example,

The Equation $x^3 * - \frac{4}{3}x - \frac{146}{27} = 0$, is firſt reduced to this Form $3x^3 * - 4x - \frac{146}{9} = 0$, and then transformed into $y^3 * - 12y - 146 = 0$.

506. Sometimes, by theſe Transformations, *Surds are taken away*. As for Example,

The Equation $x^3 - p\sqrt{a} \times x^2 + qx - r\sqrt{a} = 0$, by putting $y = \sqrt{a} \times x$, or $x = \frac{y}{\sqrt{a}}$, is transformed into this Equation, $\frac{y^3}{a\sqrt{a}} - p\sqrt{a} \times \frac{y^2}{a} + q \times \frac{y}{\sqrt{a}} - r\sqrt{a} = 0$. Which by multiplying all the Terms by $a\sqrt{a}$, becomes $y^3 - pay^2 + qay - ra^2 = 0$, an Equation free of Surds. But in order to make this ſucceed, the Surd $(\sqrt{a})$ muſt enter the alternate Terms beginning with the ſecond.

507. An Equation, as $x^3 - px^2 + qx - r = 0$, *may be transformed into one whoſe Roots ſhall be the Quantities reciprocal of* x; by ſuppoſing $y = \frac{1}{x}$, and $y = \frac{z}{r}$, or, (by one Suppoſition) $x = \frac{r}{z}$, becomes $z^3 - qz^2 + prz - r^2 = 0$.

In the Equation of y, it is manifeſt that the Order of the Coefficients is inverted; ſo that if the ſecond Term had been wanting in the propoſed Equation, the laſt but one ſhould have been wanting in the Equations of y and z. If the third had been wanting in the Equation propoſed, the laſt but two had been wanting in the Equations of y and z.

508. Another Uſe of this Tranformation is, that *the greateſt Root in the one is transformed into the leaſt Root in the other*. For ſince $x = \frac{1}{y}$, and $y = \frac{1}{x}$, it is plain that when the Value of x is greateſt, the Value of y is leaſt, and converſely.

How an Equation is transformed ſo as to have all its Ro affirmative, ſhall be explained in the following Chapter.

CHAP. XXIV.

Of finding the Roots of Equations when two more of the Roots are equal to each other.

509. BEFORE we proceed to explain how to reſolve Equ tions of all Sorts, we ſhall firſt demonſtrate *b an Equation that has two or more Roots equal, is depreſſed to a lo Dimenſion*; and its Reſolution made, conſequently, more ea And ſhall endeavour to explain the Grounds of this and ma other Rules we ſhall give in the remaining Part of this Tr tiſe, in a more ſimple and conciſe Manner than has hithe been done.

In order to this, we muſt look back to (Inſt. 495.) wh we find that if any Equation, as $x^3 - px^2 + qx - r =$ is propoſed, and you are to transform it into another that ſh have its Roots leſs than the Values of x by any given Dif rence, as e, you are to aſſume $y = x - e$, and ſubſtituting x its Value $y + e$, you find the transformed Equation,

$$\left.\begin{array}{llll} y^3 & + 3ey^2 & + 3e^2y & + e^3 \\ & - py^2 & - 2pey & - pe^2 \\ & & + qy & + qe \\ & & & - r \end{array}\right\} = 0.$$

510. Here we are to obſerve,

1°. That the laſt Term $(e^3 - pe^2 + qe - r)$ is the ve Equation that was propoſed, having e in Place of x.

2°. The Coefficient of the laſt Term but one is $3e^2 - 2$ $+ q$, which is the Quantity that ariſes by multiplying eve Term of the laſt Co-efficient $e^3 - pe^2 + qe - r$ by the I dex of e in each Term, and dividing the Product $3e^3 - 2p$ $+ qe$ by the Quantity e that is common to all the Terms

3°. T

°. The Coefficient of the laſt Term but two is $3e - p$, ch is the Quantity that ariſes by multiplying every Term of Coefficient laſt found $(3e^2 - 2pe + q)$ by the Index of each Term, and dividing the whole by $2e$.

11. Theſe ſame Obſervations extend to Equations of all nenſions. If it is the Biquadratic $x^4 - px^3 + qx^2 - rx$ $= 0$ that is propoſed, then by ſuppoſing $y = x - e$, it will ranſformed into this other,

$$\left.\begin{array}{r} y^4 + 4ey^3 + 6e^2y^2 + 4e^3y + e^4 \\ - py^3 - 3pey^2 - 3pe^2y - pe^3 \\ + qy^2 + 2qey + qe^2 \\ - ry - re \\ + s \end{array}\right\} = 0.$$

ere again it is obvious, that the laſt Term is the Equation was propoſed, having e in the Place of x. That the laſt m but one has for its Coefficient the Quantity that ariſes by tiplying the Terms of the laſt Quantity by the Indices of e ach Term, and dividing the Product by e. That the Co- ient of the laſt Term but two, (*viz.* $6e^2 - 3pe + q$) is uced in the ſame Manner from the Term immediately follow- , that is, by multiplying every Term of $4e^3 - 3pe^2 +$ $e - r$ by the Index of e in that Term, and dividing the whole multiplied into the Index of y in the Term ſought, that is, $\times 2$. And the next Term is $4e - p = \frac{6e^2 \times 2 - 3pe \times 1}{3e}$.

The Demonſtration of this may eaſily be made general by the eorem for finding the Powers of a Binomial, ſince the tranſ- ned Equation conſiſts of the Powers of the Binomial $y + e$ are marked by the Indices of e in the laſt Term, multiplied by their Coefficients $1, -p, +q, -r, +s,$ &c. re- ctively.

12. From the laſt two Articles we can eaſily find the Terms the transformed Equation without any Involution. The laſt m is had by ſubſtituting e inſtead of x in the propoſed Equa- ; the next Term, by multiplying every Part of that laſt m by the Index of e in each Part, and dividing the whole by nd the following Terms in the Manner deſcribed in the going Article; the reſpective Diviſors being the Quantity e tiplied by the Index of y in each Term.

513. The Demonſtration for finding when two or more Root are equal, will be eaſy, if we add to this, that *when the unknow Quantity enters all the Terms of any Equation, then one of its Valu is equal to nothing.*" As in the Equation $x^3 - px^2 + qx = 0$ where $x - 0 = 0$ being one of the ſimple Equations that produce $x^3 - px^2 + qx = 0$, it follows that one of the Value of x is 0. In like Manner two of the Values of x are equal t nothing in this Equation $x^3 - px^2 = 0$; and three of them vaniſh in the Equation $x^4 - px^3 = 0$.

It is alſo obvious (*converſely*) that "if, x does not enter a the Terms of the Equation, *i. e.* if the laſt Term be not wanting, then none of the Values of x can be equal to nothing," for if every Term be not multiplied by x, then $x - 0$ canno be a Diviſor of the whole Equation, and conſequently 0 canno be one of the Values of x. If x^2 does not enter into all th Terms of the Equation, then two of the Values of x canno be equal to nothing. If x^3 does not enter into all the Term of the Equation, then three of the Values of x cannot be equa to nothing, &c.

514. Suppoſe now that two Values of x are equal to on another, and to e; then it is plain that two Values of y in th transformed Equation will be equal to nothing: Since $y = x -$ e. And conſequently, by the laſt Article, the two laſt Term of the transformed Equation muſt vaniſh.

Suppoſe it is the Cubic Equation of Inſt. 509. that is propo ed, *viz.* $x^3 - px^2 + qx - r = 0$; and becauſe we ſuppo $x = e$, therefore the laſt Term of the transformed Equatio *viz.* $e^3 - pe^2 + qe - r$ will vaniſh.* And ſince two Valu of y vaniſh, the laſt Term but one, *viz.* $3e^2y - 2pey +$ will vaniſh at the ſame Time. So that $3e^2 - 2pe + q =$ But, by Suppoſition, $e = x$; therefore, when two Values of in the Equation $x^3 - px^2 + qx - r = 0$, are equal, it fo lows, that $3x^2 - 2px + q = 0$. And thus "the propoſ Cubic is depreſſed to a Quadratic that has one of its Roots equ to one of the Roots of that Cubic."

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* Becauſe ſince $x = e$, therefore $y = x - e = 0$, and conſequen all the Terms in which y is found will vaniſh; and they are all the laſt, therefore the laſt Term $e^3 - pe^2 + qe - r$ will be equal to nothing, ſince the whole Equation was ſo at firſt.

If it is the Biquadratic that is propoſed, *viz.* $x^4 - px^3 + qx^2 - rx + s = 0$, and two of its Roots be equal; then ſuppoſing $e = x$, two of the Values of y muſt vaniſh, and the Equation of Inſt. 511. will be reduced to this Form,

$$\left.\begin{matrix} y^4 + 4ey^3 + 6e^2y^2 \\ \quad - py^3 - 3pey^2 \\ \qquad\quad + qy^2 \end{matrix}\right\} ** = 0. \text{ So that}$$

$4e^3 - 3pe^2 + 2qe - r = 0$; or becauſe $x = e$,
$4x^3 - 3px^2 + 2qx - r = 0$.

515. In general, when two Values of x are equal to each other, nd to e, the two laſt Terms of the transformed Equation vaniſh: And conſequently, "if you multiply the Terms of the propoſed Equation by the Indices of x in each Term, the Quantity that will ariſe will be $= 0$, and will give an Equation of a lower Dimenſion than the propoſed, that ſhall have one of its Roots equal to one of the Roots of the propoſed Equation."

That the two laſt Terms of the Equation vaniſh when the Vaues of x are ſuppoſed equal to each other, and to e, will alſo ppear by conſidering, that ſince two Values of y then become qual to nothing, the Product of the Values of y muſt vaniſh, hich is equal to the laſt Term of the Equation; and becauſe wo of the four Values of y are equal to nothing, it follows alſo, at one of any three that can be taken out of theſe four muſt $= 0$; and therefore, the Products made by multiplying any ree muſt vaniſh; and conſequently the Coefficient of the laſt erm but one, which is equal to the Sum of theſe Products, uſt vaniſh.

516. After the ſame Manner, if there are three equal Roots the Biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$, and if be equal to one of them; three Values of y $(= x - e)$ will aniſh, and conſequently y^3 will enter all the Terms of the nsformed Equation; which will have this Form,

$$\left.\begin{matrix} y^4 + 4ey^3 \\ \quad - py^3 \end{matrix}\right\} *** = 0. \text{ So that here}$$

$6e^2 - 3pe + q = 0$; or, ſince $e = x$, therefore,
$6x^2 - 3px + q = 0$: And one of the Roots of this Quaatic will be equal to one of the Roots of the propoſed Biquaatic.

In

In this Caſe, two of the Roots of the Cubic Equation $4x^3 -$ $3px^2 + 2qx - r = 0$ are Roots of the propoſed Biquadratic, becauſe the Quantity $6x^2 - 3px + q$ is deduced from $4x^3 - 3px^2 + 2qx - r$, by multiplying the Terms by th Indexes of x in each Term.

In general, " whatever is the Number of equal Roots in th propoſed Equation, they will all remain but one in the Equation that is deduced from it, by multiplying all the Term by the Indexes of x in them; and they will all remain bu two in the Equation deduced in the ſame Manner from That; and ſo of the reſt.

517. What we obſerved of the Coefficients of Equation transformed by ſuppoſing $y = x - e$, leads to this eaſy Demonſtration of this Rule; and will be applied, in the next Chapter, to demonſtrate the Rules for finding the Limits of Equations.

It is obvious however, that tho' we make Uſe of Equatio whoſe Signs change alternately, the ſame Reaſoning extends all other Equations.

518. It is a Conſequence alſo of what has been demonſtrated, that " if two Roots of any Equation, as,

$$x^3 - px^2 + qx - r = 0, \text{ are equal,}$$

then multiplying the Terms by any arithmetical Series, as,

$$a + 3b,\ a + 2b,\ a + b,\ a, \text{ the Product will be} = 0."$$

For ſince

$$\overline{ax^3 - apx^2} + aqx - ar = 0; \text{ and}$$
$$\overline{3x^2 - 2px + q} \times bx = 0, \text{ it follows, that}$$
$$ax^3 + 3bx^3 - apx^2 - 2bpx^2 + aqx + bqx - ar =$$

Which is the Product that ariſes by multiplying the Terms the propoſed Equation by the Terms of the Series, $a +$ $a + 2b$, $a + b$, a; which may repreſent any Arithmet Progreſſion,

CHAP. XXV.

Of the Limits of Equations.

519. WE now proceed to ſhew how to diſcover the Limits of the Roots of Equations, by which their Solution is much facilitated.

Let any Equation, as $x^3 - px^2 + qx - r = 0$ be propoſed; and transform it, as above, into the Equation

$$\left.\begin{array}{rrrr} y^3 + 3ey^2 & + 3e^2y & + e^3 & \\ -py^2 & -2pey & -pe^2 & \\ & +qy & +qe & \\ & & -r & \end{array}\right\} = 0.$$

Where the Values of y are leſs than the reſpective Values of x by the Difference e. If you ſuppoſe e to be taken ſuch as to make all the Coefficients, of the Equation of y, poſitive, *viz.* $e^3 - pe^2 + qe - r$, $3e^2 - 2pe + q$, $3e - p$; then there being no Variation of the Signs in the Equation, all the Values of y muſt be negative; and conſequently, the Quantity e, by which the Values of x are diminiſhed, muſt be greater than the greateſt poſitive Value of x: And conſequently muſt be the Limit of the Roots of the Equation $x^3 - px^2 + qx - r = 0$.

It is ſufficient therefore, in order to find the Limit, to "enquire what Quantity ſubſtituted for x in each of theſe Expreſſions $x^3 - px^2 + qx - r$, $3x^2 - 2px + q$, $3x - p$, will give them all poſitive;" for that Quantity will be the Limit required.

How theſe Expreſſions are formed from one another, was explained in the Beginning of the laſt Chapter.

EXAMPLE.

520. If the Equation $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x + 120 = 0$ is propoſed; and it is required to determine the Limit that is greater than any of the Roots; you are to enquire what integer Number ſubſtituted for x in the propoſed Equation, and following Equations deduced from it by Inſt. 512, will give, in each, a poſitive Quantity,

$$\begin{array}{l} 5x^4 - 8x^3 - 30x^2 + 60x + 63 \\ 5x^3 - 6x^2 - 15x + 15 \\ 5x^2 - 4x - 5 \\ 5x - 2. \end{array}$$

The leaſt integer Number which gives each of theſe poſitive is 2; which therefore is the Limit of the Roots of the propoſed Equation; or a Number that exceeds the greateſt poſitive Root.

521. If the Limit of the *negative* Roots is required, you may (by Inſt. 494.) change the negative into poſitive Roots, and then proceed as before to find their Limits. Thus, in the Example, you will find that -3 is the Limit of the negative Roots. So that the five Roots of the propoſed Equation are betwixt -3 and $+2$.

522. Having found the Limit that ſurpaſſes the greateſt poſitive Root, call it m. And if you aſſume $y = m - x$, and for x ſubſtitute $m - y$, the Equation that will ariſe will have all its Roots poſitive; becauſe m is ſuppoſed to ſurpaſs all the Values of x, and conſequently $m - x$ $(= y)$ muſt always be affirmative. And by this Means, *any Equation may be changed into one that ſhall have all its Roots affirmative.*

Or if $-n$ repreſent the Limit of the negative Roots, then by aſſuming $y = x + n$, the propoſed Equation ſhall be transformed into one that ſhall have all its Roots affirmative; for $+n$ being greater than any negative Value of x, it follows, that $y = x + n$ muſt be always poſitive.

523. *The greateſt negative Coefficient of any Equation increaſed by Unit, always exceeds the greateſt Root of the Equation.*

To demonſtrate this, let the Cubic $x^3 - px^2 - qx - r = 0$ be propoſed; where all the Terms are negative except the firſt. Aſſuming $y = x - e$ it will be transformed into the following Equation,

$$(A)\quad \left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ -py^2 - 2pey - pe^2 \\ -qy - qe \\ -r \end{array}\right\} = 0;$$

1°. Let us ſuppoſe that the Coefficients p, q, r, are equal to each other; and if you alſo ſuppoſe $e = p + 1$, then the laſt Equation becomes

(B) $y^3 +$

$$(B)\ \left.\begin{array}{l} y^3 + 2py^2 + p^2y + 1 \\ \quad + 3y^2 + 3py \\ \qquad + 3y \end{array}\right\} = 0.$$

Where all the Terms being poſitive, it follows, that the Values of y are all negative, and that conſequently e, or $p + 1$, is greater than the greateſt Value of x in the propoſed Equation.

2°. If q and r be not $= p$, but leſs than it, and for e you ſtill ſubſtitute $p + 1$ (ſince the negative Part $\left(\begin{array}{r} -qy - qe \\ -r \end{array}\right)$ becomes leſs, the poſitive remaining undiminiſhed,) all the Coefficients of the Equation (A) become poſitive. And the ſame is obvious if q and r have poſitive Signs, and not negative Signs, as we ſuppoſed. It appears therefore, "that, if, in any Cubic Equation, p be the greateſt negative Coefficient, then $p + 1$ muſt ſurpaſs the greateſt Value of x."

524. 3°. By the ſame Reaſoning it appears, that if q be the greateſt negative Coefficient of the Equation, and $e = q + 1$, then there will be no Variation of the Signs in the Equation of y: For it appears from the laſt Article, that if all the three $(p, q, r,)$ were equal to one another, and e equal to any one of them increaſed by Unit, as to $q + 1$, then all the Terms of the Equation (A) would be poſitive. Now if e be ſuppoſed ſtill equal to $q + 1$, and p and r to be leſs than q, then all theſe Terms will be poſitive, the negative Part, which involves p and r being diminiſhed, while the poſitive Part and the negative involving q remain as before.

4°. After the ſame Manner it is demonſtrated that if r is the greateſt negative Coefficient in the Equation, and e is ſuppoſed $= r + 1$, then all the Terms of the Equation (A) of y will be poſitive; and conſequently $r + 1$ will be greater than any of the Values of x.

What we have ſaid of the *Cubic* Equation $x^3 - px^2 + qx - r = 0$, is eaſily applicable to others.

525. *In general*, we conclude, that "the greateſt negative Coefficient in any Equation increaſed by Unit, is always a Limit that exceeds all the Roots of that Equation."

But it is to be obſerved at the ſame Time, that the greateſt negative Coefficient increaſed by Unit, is very ſeldom the *neareſt* Limit: That is beſt diſcovered by the Rule in Inſt. 519.

526. As our Plan obliges us to take in every uſeful Part o Science, but no more, we ſhall here leave the remaining Part o Mr. *Maclaurin's* Diſcourſe on the *Limits of Equation*, and proceed to the ſeveral Methods of *ſolving Equations*; of theſe w ſhall propoſe ſeveral from different Authors, ſome being preferable in one Reſpect, and ſome in another, and the Learner wil be thus enabled to chuſe that which will beſt anſwer his Purpoſ in any Caſe propoſed.

527. We ſhall begin again upon this Subject with Mr. *Maclaurin*; but becauſe his Method of ſolving Equations require the Learner to have a juſt Idea of *Commenſurable Quantities* (fo his Method extends only to ſuch Equations whoſe Roots ar *commenſurable)* therefore we muſt premiſe the following Propoſition, *viz.*

528. *All Commenſurable Quantities are to each other as Numbe to Number, that is, their Proportion may be expreſſed by whole Numbers*, or they *have one common Number that will meaſure them all.*

Thus for Inſtance, let *a* and *b* be two Commenſurable Quantities, then their Proportion, be it what it will, may be expreſſe in *whole Numbers*; for let *c* be their common Meaſure withou Remainder, and let it meaſure *a* juſt three Times, and *b* fou Times; then will $3c = a$, and $4c = b$; therefore $a : b :: (3$ $: 4c ::)$ $3 : 4$ *Q.E.D.*

529. Hence *(vice verſa)* all Quantities that are to one another as Number to Number, are *Commenſurable*. Hence alſ all *whole Numbers* are commenſurable, ſince Unity is a commo Meaſure to them all.

530. Likewiſe all *Fractions* are *commenſurable*, as $\frac{a}{b}$ and $\frac{c}{d}$; f if they are reduced to a common Denomination, they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$, and $\frac{1}{bd}$ will meaſure them both.

531. *Incommenſurable Quantities* are ſuch whoſe Proportio cannot be expreſſed in whole Numbers or finite Fractions, or the can have no common Meaſure, but their Relation muſt be expreſſed by a *Surd Quantity* or *infinite Series*. Thus the Ratio of 1 $\sqrt{2}$, cannot be expreſſed in Numbers, nor the Diameter (*d*) the Circumference of a Circle (*c*) without an infinite Series, we ſhall hereafter ſee.

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f the Reſolution of Equations, all whoſe Roots are commenſurate.

2. IT was demonſtrated, in Inſt. 482. that the laſt Term of any Equation is the Product of its Roots: From which follows, that the Roots of an Equation, when commenſura- Quantities, will be found among the Diviſors of the laſt erm. And hence we have for the Reſolution of Equations, is

RULE.

Bring all the Terms to one Side of the Equation, find all the Divi- s of the laſt Term, and ſubſtitute them ſucceſſively for the unknown uantity in the Equation. So ſhall that Diviſor which, ſubſtituted this Manner, gives the Reſult = 0, be the Root of the propoſed uation.

For Example, ſuppoſe this Equation is to be reſolved,

$$\left.\begin{array}{l} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ \quad - bx^2 + 3abx \end{array}\right\} = 0,$$

ere the laſt Term is $2a^2b$, whoſe ſimple literal Diviſors are b, $2a$, $2b$, each of which may be taken either poſitively or gatively. But as here we find there are Variations of Signs in Equation, we need only take them poſitively. Suppoſe x a the firſt of the Diviſors, and ſubſtituting a for x, the E- ation becomes

$$\left.\begin{array}{l} -3a^3 + 2a^3 - 2a^2b \\ -a^2b + 3a^2b \end{array}\right\} \text{or, } 3a^3 - 3a^3 + 3a^2b - 3a^2b = 0.$$

that the whole vaniſhing, it follows that a is one of the Roots the Equation.

533. After the ſame Manner, if you ſubſtitute b in the Place x, the Equation is,

$$\left.\begin{matrix} b^3 - 3ab^2 + 2a^2b - 2a^2b \\ -b^3 + 3ab^2 \end{matrix}\right\} = 0,$$

which vaniſhing ſhews *b* to be another Root of the Equation.

Again, if you ſubſtitute $2a$ for *x*, you will find all the Te deſtroy one another ſo as to make the Sum = 0. For it then be,

$$\left.\begin{matrix} 8a^3 - 12a^3 + 4a^3 - 2a^2b \\ -4a^2b + 6a^2b \end{matrix}\right\} = 0.$$

Whence we find, that $2a$ is the third Root of the Equati Which, after the firſt two ($+a$, $+b$,) had been found, mi have been collected from this, that the laſt Term being Product of the three Roots, $+a$, $+b$ being known, th Term muſt neceſſarily be equal to the laſt Term divided by Product ab, that is, $= \frac{2a^2b}{ab} = 2a$.

534. Let the Roots of the Cubic Equation

$x^3 - 2x^2 - 33x + 90 = 0$ be required.

And firſt the Diviſors of 90 are found to be 1, 2, 3, 5, 9, 10, 15, 18, 30, 45, 90. If you ſubſtitute 1 for *x*, will find $x^3 - 2x^2 - 33x + 90 = 56$; ſo that one is n Root of the Equation. If you ſubſtitute 2 for *x*, the Re will be 24: but, putting $x = 3$, you have

$$x^3 - 2x^2 - 33x + 90 = 27 - 18 - 99 + 90 = 117 - 117$$

So that 3 is one of the Roots of the propoſed Equation. other affirmative Root is $+5$; and after you find it, as manifeſt from the Equation, that the other Root is negat you are not to try any more Diviſors taken poſitively, bu ſubſtitute them, negatively taken, for *x*: And thus you that -6 is the third Root. For putting $x = -6$, you ha

$$x^3 - 2x^2 - 33x + 90 = -216 - 72 + 198 + 90$$

This laſt Root might have been found by dividing th Term 90, having its Sign changed, by 15, the Product of two Roots already found.

535. When one of the Roots of an Equation is found, der to find the reſt with leſs Trouble, divide the propoſed E

n by the ſimple Equation which you are to deduce from the ot already found, and the Quotient ſhall give an Equation of Degree lower than the propoſed; whoſe Roots will give the maining Roots required.

As for Example, the Root + 3, firſt found, gave $x = 3$ or $-3 = 0$, whence dividing thus,

$$\begin{array}{rl} x-3\ ) & x^3 - 2x^2 - 33x + 90\ (\ x^2 + x - 30 \\ & \underline{x^3 - 3x^2} \\ & \qquad x^2 - 33x + 90 \\ & \qquad \underline{x^2 - 3x} \\ & \qquad\quad -30x + 90 \\ & \qquad\quad \underline{-30x + 90} \\ & \qquad\qquad 0 \quad\ 0 \end{array}$$

he Quotient ſhall give a Quadratic Equation $x^2 + x - 30$ 0, which muſt be the Product of the other two ſimple Equa- ns from which the Cubic is generated, and whoſe Roots erefore muſt be two of the Roots of that Cubic.

Now the Roots of that Quadratic Equation are eaſily found y Inſt. 339.) to be + 5 and − 6. For,

$$x^2 + x = 30$$

add $\frac{1}{4}$ . . $x^2 + x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4}$

$\surd$ . . . $x + \frac{1}{2} = \pm\sqrt{\frac{121}{4}} = \pm\frac{11}{2}$

and . . . $x = \pm\frac{11}{2} - \frac{1}{2} = +5$ or -6.

536. After the ſame Manner, if the Biquadratic $x^4 - 2x^3$ $25x^2 + 26x + 120 = 0$ is to be reſolved; by ſubſtituting Diviſors of 120 for x, you will find that + 3, one of thoſe iviſors is one of the Roots; the Subſtitution of 3 for x giving $-54 - 225 + 78 + 120 = 279 - 279 = 0$. And erefore dividing the propoſed Equation by $x - 3$, you muſt en- ire for the Roots of the Cubic $x^3 + x^2 - 22x - 40 = 0$, d finding that + 5, one of the Diviſors of 40, is one of the oots, you divide that Cubic by $x - 5$, and the Quotient gives Quadratic $x^2 + 6x + 8 = 0$, whoſe two Roots are − 2, 4. So that the four Roots of the Biquadratic are + 3, + 5, 2, − 4.

537. This Rule ſuppoſes that you can find all the Diviſors the laſt Term; which you may always do thus.

If it is a ſimple Quantity, divide it by its leaſt Diviſor that exce Unit, and the Quotient again by its leaſt Diviſor, proceeding thus you have a Quotient that is not diviſible by any Number greater th Unit. This Quotient, with theſe Diviſors, are the firſt or *ſim Diviſors* of the Quantity. And the Products of the Multiplic tion of any 2, 3, 4, &c. of them are the *compound Diviſors.*

As, to find the Diviſors of 60; firſt I divide by 2, and t Quotient 30 again by 2, then the next Quotient 15 by 3, a the Quotient of this Diviſion 5 is not farther diviſible by any I teger above Units; ſo that the ſimple Diviſors are,

2, 2, 3, 5;
The Products of two, 4, 6, 10, 15.
The Products of three, 12, 20, 30.
The Product of all four, ... 60.

The Diviſors of 90 are found after the ſame Manner;

Simple Diviſors, 2, 3, 3, 5.
The Products of two, 6, 9, 10, 15.
The Products of three, 18, 30, 45.
The Product of all four, ... 90.

The Diviſors of 21 *abb*.

The ſimple Diviſors, 3, 7, *a*, *b*, *b*.
The Products of two, 21, 3*a*, 3*b*, 7*a*, 7*b*, *ab*, *bb*.
The Products of three, 21*a*, 21*b*, 3*ab*, 3*bb*, 7*ab*, 7*bb*, *a*
The Products of four, 21*ab*, 21*bb*, 3*abb*, 7*abb*.
The Products of the five, ... 21*a*

538. But as the laſt Term may have very many Diviſors, the Labour may be very great to ſubſtitute them all for the u known Quantity, we ſhall now ſhew how it may be abridg by limiting to a ſmall Number the Diviſors you are to try. An firſt it is plain, (from Inſt. 523.) that "any Diviſor that ceeds the greateſt negative Coefficient by Unity is to be negle

ed." Thus in reſolving the Equation $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$, as 25 is the greateſt negative Coefficient, we conclude that the Diviſors of 120 that exceed 26 may be neglected.

539. But the Labour may be ſtill abridged, if we make Uſe of the Rule in Inſt. 519. that is, if we find the Number which ſubſtituted in theſe following Expreſſions,

$$x^4 - 2x^3 - 25x^2 + 26x + 120,$$
$$2x^3 - 3x^2 - 25x + 13,$$
$$6x^2 - 6x - 25$$
$$2x - 1,$$

will give in them all a poſitive Reſult: For that Number will be greater than the greateſt Root, and all the Diviſors of 120 that exceed it may be neglected.

That this Inveſtigation may be eaſier, we ought to begin always with that Expreſſion, where the negative Roots ſeem to prevail moſt; as here in the Quadratic Expreſſion $6x^2 - 6x - 25$; where finding that 6 ſubſtituted for x gives that Expreſſion poſitive, and gives all the other Expreſſions at the ſame Time poſitive, I conclude that 6 is greater than any of the Roots, and that all the Diviſors of 120 that exceed 6 may be neglected.

540. If the Equation $x^3 + 11x^2 + 10x - 72 = 0$ is propoſed, the Rule of Inſt. 523. does not help to abridge the Operation; the laſt Term itſelf being the greateſt negative Term. But, by Inſt. 519. we enquire what Number ſubſtituted for x will give all theſe Expreſſions poſitive.

$$x^3 + 11x^2 + 10x - 72$$
$$3x^2 + 22x + 10$$
$$3x + 11$$

Where the Labour is very ſhort, ſince we need only attend to the firſt Expreſſion; and we ſee immediately that 4 ſubſtituted for x gives a poſitive Reſult, whence all the Diviſors of 72 that exceed 4 are to be rejected; and thus, by a few Trials, we find that $+ 2$ is the poſitive Root of the Equation. Then dividing the Equation by $x - 2$, and reſolving the Quadratic Equation that is the Quotient of the Diviſion, you find the other two Roots to be $- 9$, and $- 4$.

541. Beſides the Method already explained, there are others by which Limits may be determined, which the Root of an Equation cannot exceed.

Since the Squares of all real Quantities are affirmative, it follows, that *The Sum of the Squares of the Roots of any Equation muſt be greater than the Square of the greateſt Root.* And the Square Root of that Sum will therefore be a Limit that muſt exceed the greateſt Root of the Equation.

If the Equation propoſed is $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$, &c. $= 0$, then the Sum of the Squares of the Roots (by Inſt. 485.) will be $p^2 - 2q$. So that $\sqrt[2]{p^2 - 2q}$ will exceed the greateſt Root of that Equation.

Or if you find, (by Inſt. 486.) the Sum of the 4th Powers of the Roots of the Equation, and extract the Biquadratic Root of that Sum, it will alſo exceed the greateſt Root of the Equation.

542. But there is another Method that reduces the Diviſors of the laſt Term, that can be uſeful, ſtill to more narrow Limits.

Suppoſe the Cubic Equation $x^3 - px^2 + qx - r = 0$ is propoſed to be reſolved. Transform it to an Equation whoſe Roots ſhall be leſs than the Values of x by Unity, aſſuming $y = x - 1$. And the laſt Term of the transformed Equation will be $1 - p + q - r$; which is found by ſubſtituting Unit, the Difference of x and y, for x, in the propoſed Equation; as will eaſily appear from Inſt. 495. where, when $y = x - e$, the laſt Term of the transformed Equation was $e^3 - pe^2 + qe - r$.

Transform again the Equation $x^3 - px^2 + qx - r = 0$, by aſſuming $y = x + 1$, into an Equation whoſe Roots ſhall exceed the Values of x by Unit, and the laſt Term of the tranſformed Equation will be $-1 - p - q - r$, the ſame that ariſes by ſubſtituting -1, the Difference betwixt x and y, for x in the propoſed Equation.

Now the Values of x are ſome of the Diviſors of r, which is the Term left when you ſuppoſe $x = 0$; and the Values of the y's are ſome of the Diviſors of $+1 - p + q - r$, and of $-1 - p - q - r$, reſpectively. And theſe Values are in Arithmetical Progreſſion increaſing by the common Difference Unit; becauſe $x - 1$, x, $x + 1$, are in that Progreſſion. And it is obvious the ſame Reaſoning may be extended to any Equation of whatever Degree. So that this gives a general Method for the Reſolution of Equations whoſe Roots are commenſurable.

RULE.

Subſtitute in Place of the unknown Quantity ſucceſſively the Terms of the Progreſſion 1, 0, —1, &c. *and find all the Diviſors of the Sums that reſult*; *then take out all the arithmetical Progreſſions you can find among theſe Diviſors, whoſe common Difference is Unit*; *and the Values of* x *will be among the Diviſors ariſing from the Subſtitutions of* $x = 0$ *that belong to theſe Progreſſions.* The Values of x will be affirmative when the arithmetical Progreſſion increaſes, but negative when it decreaſes.

EXAMPLE.

543. Let it be required to find one of the Roots of the Equation $x^3 - x^2 - 10x + 6 = 0$. The Operation is thus;

Suppoſit.	*Reſult.*		*Diviſors.*	*Arith. Prog. decr.*	
$x = 1$	$x^3 - x^2 - 10x + 6 =$	-4	1,2,4	4	
$x = 0$		$+6$	1,2,3, 6	3	gives $x = -3$
$x = -1$		$+14$	1,2,7,14	2	

Where the Suppoſitions of $x = 1$, $x = 0$, $x = -1$ give the Quantity $x^3 - x^2 - 10x + 6$ equal to -4, 6, 14; among whoſe Diviſors we find only one arithmetical Progreſſion 4, 3, 2; the Term of which oppoſite to the Suppoſition of $x = 0$, being 3, and the Series decreaſing, we try if -3 ſubſtituted for x makes the Equation vaniſh; which ſucceeding one of its Roots muſt be -3. Then dividing the Equation by $x + 3$, we find the Roots of the (Quadratic) Quotient
$x^2 - 4x + 2 = 0$ are $2 \pm \sqrt{2}$.

544. If it is required to find the Roots of the Equation $x^3 - 3x^2 - 46x - 72 = 0$, the Operation will be thus;

Suppoſ.	Reſults	Diviſors.	Progreſſions.			
$x = 1$	-120	1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120.	8	3	4	5
$x = 0$	-72	1,2,3,4,6,8,9,12,18,24,36,72.	9	2	3	4
$x = -1$	-30	1,2,3,5,6,10,15,30.	10	1	2	3

Of theſe four Arithmetical Progreſſions having their co mon Difference equal to Unit, the firſt gives $x = 9$, the oth give $x = -2$, $x = -3$, $x = -4$; all which ſucceed exc $x = -3$: So that the three Values of x are $+ 9$, $- 2$, —

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f the Reſolution of Equations by Cardan's *Rule*,† *and others of that Kind.*

. THE ſecond Term can be taken away out of any *Cubic* Equation, by Inſt. 497; ſo that they all may be ıced to this Form,

$$x^3 * + qx + r = 0.$$

et us ſuppoſe that $x = a + b$; and $x^3 + qx + r = a^3$ $a^2 b + 3ab^2 + b^3 + qx + r = a^3 + 3ab \times \overline{a+b} +$ $+ qx + r = a^3 + 3abx + b^3 + qx + r =$ (by ſupng $3ab = -q$) $= a^3 + b^3 + r = 0$.

ut $b = -\frac{q}{3a}$, and $b^3 = -\frac{q^3}{27a^3}$ and conſequently, a^3 $\frac{q^3}{27a^3} + r = 0$; or, $a^6 + ra^3 = \frac{q^3}{27}$.

ıppoſe $a^3 = z$, and you have, $z^2 + rz = \frac{q^3}{27}$; which is a dratic whoſe Reſolution gives

$$z = -\tfrac{1}{2}r \pm \sqrt{\tfrac{1}{4}r^2 + \frac{q^3}{27}} = a^3,$$

$$\text{and } a = \sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r^2 + \frac{q^3}{27}}}; \text{ and}$$

$$a + b = a - \frac{q}{3a} = \sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r^2 + \frac{q^3}{27}}} - \frac{q}{\sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r + \frac{q^3}{27}}}}$$

: in which Expreſſions there ıly known Quantities.

546. The

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ho' this Method be generally aſcribed to *Cardan*, yet *Cardan* f mentions it as the Invention of one *Scipio Ferreus*, a noted ematician before his Time.

546. The Values of x may be found a little differently thus;

Since $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, it follows,

that $a^3 + r = +\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, and

$b^3\ (= -a^3 - r) = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$; ſo that $b =$

$\sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; and $x\ (= a + b) =$

$$\sqrt[3]{-\frac{1}{2}r \pm \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$$

which gives but one Value of x, becauſe when, in the Value a the Surd $\sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ is poſitive, it is negative in the Value of b, and there is only the Difference of this Sign in the Values. So that we may conclude

$$x = \sqrt[3]{-\frac{1}{2}r + \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt[2]{\frac{1}{4}r^2 + \frac{q}{2}}}$$

547. * The Values of x may be diſcovered without exterminating the ſecond Term.

Any Cubic Equation may be reduced to this Form,

$$\left.\begin{array}{l} x^3 - 3px^2 - 3qx - 2r \\ \qquad\quad + 3p^2x - p^3 \\ \qquad\qquad\quad + 3pq \end{array}\right\} = 0,$$

which by ſuppoſing $x = z + p$, will be reduced to $z^3 * -3$ $-2r = 0$, in which the ſecond Term is wanting. But by t laſt Article, ſince $z^3 * - 3qz - 2r = 0$, it follows that

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* This Method we owe to the Reverend and very Learned M John Colson, the preſent *Lucaſian* Profeſſor of Mathematics in Univerſity of Cambridge, and Succeſſor to Dr. *Saunderſon.* Philoſ. Tranſ. 309.

$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}} =$ (if you ſuppoſe that the Cubic Root of the Binomial $r + \sqrt{r^2 - q^3}$ is $m + \sqrt{n}$) $= m + \sqrt{n} + m - \sqrt{n} = 2m$. And ſince $x = z + p$, it follows that $x = p + 2m$.

548. But as the ſquare Root of any Quantity is *two-fold*, "the Cube Root is *three-fold*," and can be expreſſed three different Ways.

Suppoſe the Cube Root of Unit is required, and let $y^3 = 1$, or $y^3 - 1 = 0$, then ſince Unit itſelf is a Cube Root of 1, one of the Values of y is 1, ſo that the Equation $y - 1 = 0$ ſhall divide the firſt Equation $y^3 - 1 = 0$, and the Quotient $y^2 + y + 1 = 0$, reſolved, gives $y = \frac{-1 \mp \sqrt{-3}}{2}$; ſo that the 3 Expreſſions of $\sqrt[3]{1}$ are 1, $\frac{-1 + \sqrt{-3}}{2}$, and $\frac{-1 - \sqrt{-3}}{2}$.

And, in general, the Cube Root of any Quantity A^3† may be A, or $\frac{-1 + \sqrt{-3}}{2} \times A$, or $\frac{-1 - \sqrt{-3}}{2} \times A$; ſo that the Cube Root of the Binomial $r + \sqrt{r^2 - q^3}$ may be $m + \sqrt[2]{n}$, as we ſuppoſed above, or $\frac{-1 + \sqrt{-3}}{2} \times \overline{m + \sqrt[2]{n}}$, or $\frac{-1 - \sqrt{-3}}{2} \times \overline{m + \sqrt[2]{n}}$. And hence we have three Expreſſions for x, *viz.*

1. $x = p + 2m$,
2. $x = p - m + \sqrt{-3n}$.
3. $x = p - m - \sqrt{-3n}$.

and theſe give the three Roots of the propoſed Cubic Equation.

EXAMPLE I.

549. Let it be required to find the Roots of the Equation $x^3 - 12x^2 + 41x - 42 = 0$.

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† Since $A^3 = A^3 \times 1$, and $\sqrt[3]{A^3} \times \sqrt[3]{1} = A$.

Comparing the Coefficients of this Equation with thoſe of th general Equation

$$x^3 - 3p\,x^2 \left.\begin{array}{l} -3q \\ +3p^2 \end{array}\right\} x \left.\begin{array}{l} -2r \\ -p^3 \\ +3pq \end{array}\right\} = 0,$$ you find,

1. $3p = 12$, ſo that . . . : . . . . . . $p = 4$
2. $3p^2 - 3q\,(= 48 - 3q) = 41$ . . . . . $q =$
3. $3pq - p^3 - 2r\,(= -36 - 2r) = -42$. $r = 3$

and conſequently, $r^2 - q^3 = 9 - \frac{343}{27} = -\frac{100}{27}$, and $r +$ $\sqrt{r^2 - q^3} = 3 + \sqrt{-\frac{100}{27}}$. Now the Cube Root of thi Binomial is found to be, $-1 + \sqrt[2]{-\frac{4}{3}}$ $(= m + \sqrt{n}.)$ Whence

1. $x = p + 2m = 4 - 2 = 2$.
2. $x = p - m - \sqrt{-3n} = 4 + 1 - \sqrt{4} = 5 - 2 = 3$.
3. $x = p - m + \sqrt{-3n} = 5 + 2 = 7$.

So that the three Roots of the propoſed Equation are 2, 3, 7.

You may find other two Expreſſions of the Cube Root of $+ \sqrt{-\frac{100}{27}}$, beſides $-1 + \sqrt{-\frac{4}{3}}$, *viz.* $\frac{3}{2} + \sqrt{-\frac{1}{1}}$ and $-\frac{1}{2} - \sqrt{-\frac{25}{12}}$; but theſe ſubſtituted for $m + \sqrt{n}$ giv the ſame Values for x as are already found.

EXAMPLE II.

550. In the Equation $x^3 + 15x^2 + 84x - 100 = 0$, yo find $p = -5$, $q = -3$, $r = 135$, and $r + \sqrt{r^2 - q^3} = 13$ $+ \sqrt{18252}$, whoſe Cube Root is $3 + \sqrt{12}$; ſo that $x = ($ $+ 2m) = -5 + 6 = 1$. The other two Values of x, *vi* $-8 + \sqrt{-36}$, $-8 - \sqrt{-36}$, are impoſſible. For i this Caſe $n = 12$, therefore $-3n = -36$, which makes th Quantity $\sqrt{-3n}$, and of Courſe the two laſt Roots, impoſſible

After the ſame Manner you will find that the Roots of th Equation $x^3 + x^2 - 166x + 660 = 0$, are -15, $7 \pm \sqrt{5}$

An

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† We ſhall, hereafter, ſhew, from Mr. *De Moivre*, how the Cubi Root of an impoſſible Binomial, as $a + \sqrt{-b}$, may be extracte by the *Triſection* of an Angle.

And here we are to obſerve, that whenever this irrational Part, $\sqrt{r^2 - q^3}$, is impoſſible, that is, as often as q is an affirmative Quantity, and at the ſame Time its Cube is greater than r^2, the three Roots of the Equation are all poſſible and real; but if $\sqrt{r^2 - q^3}$ be poſſible, that is, if q be negative, or if affirmative, its Cube be leſs than r^2, the Equation will have but one poſſible and real Root, the other two being impoſſible or imaginary. This is eaſily deduced from the Structure of the Equation, as we have juſt now ſhewn.

Note, If in this general Theorem we put $p = 0$, the ſecond Term vaniſhes, and the Equation is reduced to *Cardan*'s Forms before ſpecified.

551. The Roots of Biquadratic Equations may be found by reducing them to Cubes, thus.

Let the ſecond Term be taken away by the Rule given in Inſt. . and let the Equation that reſults be,

$$x^4 * + qx^2 + rx + s = 0.$$

And let us ſuppoſe this Biquadratic to be the Product of theſe two Quadratic Equations,

$$\begin{array}{l} x^2 + ex + f = 0 \\ x^2 - ex + g = 0 \\ \hline \left.\begin{array}{l} x^4 * +f \\ \quad\;\; +g \\ \quad\;\; -e^2 \end{array}\right\} \times x^2 \left.\begin{array}{l} +eg \\ -ef \end{array}\right\} \times x + fg = 0. \end{array}$$

Where e is the Coefficient of x in both Equations but effected with contrary Signs; becauſe when the ſecond Term is wanting in an Equation, the Sum of the affirmative Roots muſt be equal to the Sum of the negative.

Compare now the propoſed Equation with the above Product, and the reſpective Terms put equal to each other will give $f + g - e^2 = q$, $eg - ef = r$, $fg = s$. Whence it follows that $f + g = q + e^2$, and $g - f = \frac{r}{e}$; and conſequently,

$f + g + g - f\ (= 2g) = q + e^2 + \frac{r}{e}$, and

$g =$

$g = \frac{q+e^2+\frac{r}{e}}{2}$; the ſame Way, you will find, by Sub-traction, &c. $f = \frac{q+e^2-\frac{r}{e}}{2}$, and $f \times g\ (= s) = \frac{1}{4} \times$ $\overline{q^2+2qe^2+e^4-\frac{r^2}{e^2}}$; and multiplying by $4e^2$ and rangin the Terms, you have this Equation,

$$e^6+2qe^4+\overline{q^2-4s}\times e^2-r^2=0.$$

Suppoſe $e^2=y$, and it becomes $y^3+2qy^2+\overline{q^2-4s}\,y-$ $r^2=0$, a Cubic Equation whoſe Roots are to be diſcovered b the preceding Articles. Then the Values of y being found their Square Roots will give e (ſince $y=e^2$;) and having you will find f and g from the Equations $f=\frac{q+e^2-\frac{r}{e}}{2}$, $=\frac{q+e^2+\frac{r}{e}}{2}$. Laſtly extracting the Roots of the Equation $x^2+ex+f=0$, $x^2-ex+g=0$, you will find the fo Roots of the Biquadratic $x^4 * qx^2+rx+s=0$; for eith $x=-\frac{1}{2}e\pm\sqrt{\frac{1}{4}e^2-f}$, or, $x=+\frac{1}{2}e\pm\sqrt{\frac{1}{4}e^2-g}$.

552. Or if you want to find the Roots of the Biquadrat without taking away the ſecond Term;

Suppoſe it to be of this Form,

$$x^4-4px^3\left.\begin{matrix}-2q\\+4p^2\end{matrix}\right\}x^2\left.\begin{matrix}-8r\\+4pq\end{matrix}\right\}x\left.\begin{matrix}-4s\\+q^2\end{matrix}\right\}=0,$$

and the Values of x will be

$$\left.\begin{matrix}x=p-a\pm\sqrt{p^2+q-a^2-\frac{2r}{a}}\\ \text{and } x=p+a\pm\sqrt{p^2+q-a^2+\frac{2r}{a}}\end{matrix}\right\}\text{where}$$

a^2 is equal to the Root of the Cubic,

$$y^3 \begin{matrix} -p^2 \\ -q \end{matrix}\Big\} y^2 \begin{matrix} +2pr \\ +s \end{matrix}\Big\} y - r^2 = 0.$$

The Demonſtration is deduced from the laſt Article, as that of 546 is from the preceding.

553. An ingenious Correſpondent, Mr. **Samuel Cole**, who teaches a Mathematical School near *Horndean* in *Hampſhire*, has obliged me with the following plain Demonſtration of *Cardan's Rules*, from their firſt Principles, as alſo an Improvement of the ſame, which here follow.

" There are two Forms of Cubic Equations, *viz.*

1. $x^3 + px = \pm q,$
2. $x^3 - px = \pm q.$

554.

CASE I.

	1	$x^3 + px = +q.$ Query x?
Suppoſe	2	$x = m - n.$ Becauſe x is here affirmative.
$2 @^{3}$	3	$x^3 = m^3 - 3m^2n + 3mn^2 - n^3$
$2 \times \bar{p}$	4	$px = pm - pn$
$3 + 4$	5	$\left\{\begin{matrix} x^3 + px = m^3 - 3m^2n + 3mn^2 - n^3 \\ + pm - pn = q. \end{matrix}\right.$
Again, Suppoſe	6	$3mn = p.$ (Or, $p - 3mn = 0$) [Alſo $n = \frac{\frac{1}{3}p}{m}$
Then $6 \times \overline{m - n}$	7	$3mmn - 3mnn = pm - pn$
equated to (0)	8	$0 = -3m^2n + 3mn^2 + pm - pn$
$5 - 8$	9	$(x^3 + px =)\ m^3 - n^3 = q$
But	10	$3mn = p.$ By Suppoſition, Step 6th.
$10 \div 3m$	11	$n = \frac{p}{3m}$
$11 @^{3}$	12	$n^3 = \frac{p^3}{27m^3}.$ (And $-n^3 = -\frac{p^3}{27m^3}$)
9, 12	13	$\left\{\begin{matrix} m^3 - (n^3 = \frac{p^3}{27m^3} = q. \text{ (Or, } m^3 - \\ \frac{p^3}{27m^3} = q.) \text{ By Subſtitution.} \end{matrix}\right.$
$13 \times m^3$	14	$m^6 - \frac{p^3}{27} = qm^3$

14 by

14 by Reduction	15	$m^6 - q m^3 = \frac{p^3}{27}$. A quadratic Equation.
15 C □	16	$m^6 - q m^3 + \frac{q^2}{4} = \frac{p^3}{27} + \frac{q^2}{4}$
16 lw^2	17	$m^3 - \frac{q}{2} = \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$
17 $+ \frac{q}{2}$	18	$m^3 = \frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ A ſimple Cubic
18 lw^3	19	$m = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$
But (9th Step)	20	$q = m^3 - n^3$
Subſt. Value of m^3 from Step 18.	21	$\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}} - n^3 = q$
21 by Reduction.	22	$-n^3 = q - \frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ Or,(becauſe $q - \frac{q}{2} = \frac{q}{2}$)
22 contracted	23	$-n^3 = \frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ A ſimp. Cubic
23 lw^3	24	$-n = -\sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$
19, 24	25	$m - n = (x =) \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$ $- \sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$ The Theor.

555. If the Equation be $x^3 + p x = -q$, find the Root of $x^3 + px = +q$, as above, and change its Sign.

556. *Cardan*'s Rule, above given, requires two Extractions of the Cube Root, and therefore the following Method, which requires but *one*, is much eaſier.

See the 19th Step above, $m = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$

And from the 6th Step it appears that

$$n = \frac{\frac{1}{3}p}{m} = \frac{\frac{1}{3}p}{\sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}}$$

Which is an Improvement of *Cardan*'s firſt Rule.

557. CASE

CASE II.

57. $x^3 - px = \pm q$. This is demonſtrated in the ſame Manner before, by putting $x = m + n$, &c. Here alſo $n = \frac{\frac{1}{3}p}{m} =$

$$\frac{\frac{1}{3}p}{\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}}$$

, and x is found by one Extraction.

CHAP. XXVIII.

f the NEWTONIAN METHOD of approximating to the Roots of NUMERICAL EQUATIONS.

58. WHEN any Equation is propoſed to be reſolved, if the Root lies between two Numbers, that differ from e another by Unity, you may conclude that Root is *incommen-rable*, by what has been heretofore taught. The Root, how-ver, may be approximated to what Degree of Exactneſs you leaſe, by the following Method, firſt propoſed by Sir *Iſaac Newton*.

559. In the firſt Place you make Trial with ſuch Numbers as ou find (by the foregoing Articles) are near the Value of the oot, and you will ſoon get two Numbers, one of which will ive the Equation a *poſitive*, and the other a *negative Value*, and onſequently the true Root, which makes it vaniſh, or = 0, ill be between theſe Numbers, and when it is incommenſurate ou may, by eaſy Trials, approach to it as near as can be neceſ-ry by Fractions either *vulgar*, or *decimal*.

560. But before we proceed any farther, it may be worth ob-rving, *that any common Number expreſſed by the nine Digits, is t a ſimple, quadratic, cubic*, &c. *Equation*, whoſe Root is 10 $= x$, and therefore not expreſſed, but only the Coefficients of ch Term. Thus, for Example,

$$\text{The Number}\left\{\begin{array}{l} 96 = 9x + 6, \text{ a \textit{Simple} Equation.}\\ 596 = 5x^2 + 9x + 6, \text{ a \textit{Quadratic.}}\\ 7596 = 7x^3 + 5x^2 + 9x + 6, \text{ a \textit{Cubic.}}\\ 17596 = x^4 + 7x^3 + 5x^2 + 9x + 6, \text{ a \textit{Biquadra}}\end{array}\right.$$

Where, for the Sake of expeditious Operation, we abbrevi thoſe Equations into the Expreſſion of their Coefficients on by calling x *Ten*, x^2 an *Hundred*, x^3 a *Thouſand*, &c. which they conſiſt of Cyphers only, with Unit prefixed, *viz.* 10, 10 1000, &c. can be eaſily underſtood without being expreſſed the common Form of an *Equation* or *Series*. But to proceed.

561. Let the Equation propoſed be $x^2 - 6x\ 7 = 0$, if ſuppoſe $x = 2$, the Reſult is $4 - 12 + 7 = -1$, which b ing negative, and the Suppoſition of $x = 0$ giving a poſiti Reſult, it follows that the Root is betwixt 0 and 2. Next, ſuppoſe $x = 1$; whence, $x^2 - 6x + 7 = 1 - 6 + 7$ $+ 2$, which being poſitive, we infer the Root is betwixt 1 a 2, and conſequently incommenſurable. In order to appro mate to it, we ſuppoſe $x = 1\frac{1}{2}$, and find $x^2 - 6x + 7 =$ $- 9 + 7 = \frac{1}{4}$; and this Reſult being poſitive, we infer t Root muſt be betwixt 2 and $1\frac{1}{2}$. And therefore we try $1\frac{3}{4}$, a find $x^2 - 6x + 7 = \frac{49}{16} - \frac{42}{4} + 7 = 3\frac{1}{16} - 10\frac{8}{16} + 7$ $- \frac{7}{16}$, which is negative; ſo that we conclude the Root to betwixt $1\frac{3}{4}$ and $1\frac{1}{2}$. And therefore we try next $1\frac{5}{8}$, which gi ing alſo a negative Reſult, we conclude the Root is betwixt (or $1\frac{4}{8}$) and $1\frac{5}{8}$. We try therefore $1\frac{9}{16}$, and the Reſult bei poſitive, we conclude that the Root muſt be betwixt $1\frac{9}{16}$ a $1\frac{10}{16}$, and therefore is nearly $1\frac{19}{32}$.

562. Or you may approximate more eaſily by transformi the Equation propoſed into another whoſe Roots ſhall be equal 10, 100, or 1000 Times the Root of the former (by Inſt. 50 and taking the Limits greater in the ſame Proportion. Th Transformation is eaſy; for you are only to multiply the Term by 10, 100, or 1000, the third Term by their Squar the 4th by their Cubes, &c. The Equation of the laſt Exa ple is thus transformed into $x^2 - 600x + 70000 = 0$, who Roots are 100 Times the Roots of the propoſed Equation, a whoſe Limits are 100 and 200. Proceeding as before, we t 150, and find $x^2 - 600x + 70000 = 22500 - 90000$ $70000 = 2500$, ſo that 150 is leſs than the Root. You ne

y 175, which, giving a negative Reſult muſt be greater than e Root: and thus proceeding you find the Root to be betwixt 8 and 159: from which you infer, that the leaſt Root of the opoſed Equation $x^2 - 6x + 7 = 0$ is betwixt 1.58 and 1.59, ing the hundredth Part of the Root of $x^2 - 600x + 70000$ 0.

563. If the Cubic Equation $x^3 - 15x^2 + 63x - 50 = 0$ propoſed to be reſolved, by ſubſtituting 0 for x the Value of $-15x^2 + 63x - 50$ is negative, and by ſubſtituting 3 for that Quantity becomes poſitive. $x = 1$ gives it negative, d $x = 2$ gives it poſitive, ſo that the Root is between 1 and 2, d therefore incommenſurable. You may proceed as in the egoing Examples to approximate to the Root. But there are er Methods by which you may do that more eaſily and rea- y; which we proceed to explain.

564. When you have diſcovered the Value of the Root to leſs n an Unit (as in this Example you know it is a little above 1) poſe the Difference betwixt its real Value and the Number t you have found nearly equal to it, to be repreſented by f: in this Example.

Let $x = 1 + f$. Subſtitute this Value for x in the Equa- n, thus,

$$\begin{array}{rcl}
x^3 & = & 1 + 3f + 3f^2 + f^3 \\
-15x^2 & = & -15 - 30f - 15f^2 \\
+63x & = & 63 + 63f \\
-50 & = & -50 \\
\hline
\end{array}$$

$$-15x^2 + 63x - 50 = -1 + 36f - 12f^2 + f^3 = 0.$$

Now becauſe f is ſuppoſed leſs than Unit, its Powers f^2, f^3, y be neglected in this Approximation; ſo that aſſuming only two firſt Terms, we have $-1 + 36f = 0$, or $f = \frac{1}{36} =$ 7; ſo that x will be nearly 1.027.

65. You may have a nearer Value of x by conſidering, that g $-1 + 36f - 12f^2 + f^3 = 0$, it follows that

$$f = \frac{1}{36 - 12f + f^2} = \text{(by ſubſtituting } \tfrac{1}{36} \text{ for } f\text{) near}$$

$$= \frac{1}{36 - 12 \times \frac{1}{36} + \frac{1}{36} \times \frac{1}{36}} = \tfrac{1296}{46225} = .02803.$$

566. But the Value of f may be corrected and determin more accurately by ſuppoſing g to be the Difference betwixt real Value, and that which we laſt found nearly equal to it. that $f = .02803 + g$. Then by ſubſtituting this Value fo in the Equation

$f^3 - 12f^2 + 36f - 1 = 0$, it will ſtand as follows,

$$\left.\begin{aligned} f^3 &= 0.0000220226 + 0.002357g + 0.08409g^2 + g^3 \\ -12f^2 &= -.00942816 \quad - 0.67272g - 12g^2 \\ +36f &= 1.00908 \quad + 36g \\ -1 &= -1. \end{aligned}\right\}$$

$$= -0.0003261374 + 35.329637g - 11.9195g^2 + g^3 =$$

Of which the two firſt Terms, neglecting the reſt, g

$35.329637 \times g = 0.0003261374$, and $g = \frac{.0003261374}{35.329637}$

0.00000923127. So that $f = 0.02803923127$; and $x =$ $+ f = 1.02803923127$; which is very near the true Root the Equation that was propoſed.

If ſtill a greater Degree of Exactneſs is required, ſuppo equal to the Difference betwixt the true Value of g and that have already found, and proceeding as above you may corr the Value of g.

567. For another Example; let the Equation to be reſol be $x^3 - 2x - 5 = 0$, and by ſome of the preceding Meth you diſcover one of the Roots to be between 2 and 3. Th fore you ſuppoſe $x = 2 + f$, and ſubſtituting this Value fo you find,

$$\left.\begin{aligned} x^3 &= 8 + 12f + 6f^2 + f^3 \\ -2x &= -4 - 2f \\ -5 &= -5 \end{aligned}\right\} = 0 =$$

$$= -1 + 10f + 6f^2 + f^3;$$

from which we find that

$10f = 1$ nearly, or $f = 0.1$. Then to correct this Value, we ſuppoſe $f = 0.1 + g$, and find

$$\left.\begin{array}{rl} f^3 = & 0.001 + 0.03g + 0.3g^2 + g^3 \\ 6f^2 = & 0.06 + 1.2\,g + 6.\,g^2 \\ 10f = & 1. + 10.\,g \\ -1 = & -1. \end{array}\right\} = 0$$

$$= 0.061 + 11.23g + 6.3g^2 + g^3,$$

$$\text{ſo that } g = \frac{-0.061}{11.23} = -0.0054.$$

Then by ſuppoſing $g = -.0054 + h$, you may correct its Value, and you will find that the Root required is nearly 2.09455147.

568. "In all theſe Operations, you will approximate ſooner to the Value of the Root, if you take the three laſt Terms of the Equation, and extract the Root of the Quadratic Equation conſiſting of theſe three Terms."

Then, in Inſt. 564, inſtead of the two laſt Terms of the Equation $f^3 - 12f^2 + 36f - 1 = 0$, if you take the three laſt and extract the Root of the Quadratic $12f^2 - 36f + 1 = 0$, you will find $f = .028031$, which is much nearer the true Value than what you diſcover by ſuppoſing $36f - 1 = 0$.

It is obvious, that this Method extends to all Equations.

569. "By aſſuming Equations affected with general Coefficients, you may, by this Method, deduce general Rules or *Theorems* for approximating to the Roots of propoſed Equations of whatever Degree."

Let $f^3 - pf^2 + qf - r = 0$ repreſent the Equation by which the Fraction f is to be determined, which is to be added to the Limit, or ſubtracted from it, in order to have the near Value of x.

Then $qf - r = 0$ will give $f = \frac{r}{q}$. But ſince $f = \frac{r}{f^2 - pf + q}$, by ſubſtituting $\frac{r}{q}$ for f, we have this Theorem for finding f nearly, *viz.*

$f =$

$$f=\frac{r}{\frac{r^2}{q^2}-\frac{pr}{q}+q}=\frac{q^2\times r}{q^3-pqr+r^2}.$$

570. After the ſame Manner, if it is a Biquadratic, by which f is to be determined, as $f^4-pf^3+qf^2-rf+s=0$, then f being very little, we ſhall have $f=\frac{s}{r}$; which Value is corrected by conſidering that $f=\frac{s}{r-qf+pf^2-f^3}=$ (by ſubſtituting $\frac{s}{r}$ for f) $=\frac{s}{r-\frac{qs}{r}+\frac{ps^2}{r^2}-\frac{s^3}{r^3}}$, whence we have this Theorem for all Biquadratic Equations,

$$f=\frac{r^3\times s}{-s^3+ps^2r-qsr^2+r^4}.$$

571. Other Theorems may be deduced by aſſuming the three Terms of the Equation, and extracting the Root of the Quadratic which they form.

Thus to find the Value of f in the Equation $f^3-pf^2+qf-r=0$ where f is ſuppoſed to be very little, we neglect the firſt Term f^3, and extract the Root of the Quadratic $pf^2-qf+r=0$, or of $f^2-\frac{q}{p}\times f+\frac{r}{p}=0$; and we find $f=\frac{q}{2p}\pm\sqrt{-\frac{r}{p}+\frac{q^2}{4p^2}}=\frac{q\pm\sqrt{q^2-4pr}}{2p}$ nearly.

But this Value of f may be corrected by ſuppoſing it equal to m, and ſubſtituting m^3 for f^3 in the Equation $f^3-pf^2+qf-r=0$, which will give $m^3-pf^2+qf-r=0$, and $pf^2-qf+r-m^3=0$; the Reſolution of which Quadratic Equation gives $f=\frac{q\pm\sqrt{q^2-4pr+4pm^3}}{2p}$, very near the true Value of f.

After the ſame Manner you may find like Theorems for the Roots of Biquadratic Equations, or of Equations of any Dimention whatever.

572. In general, let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$, &c. $+ A = 0$ repreſent an Equation of any Dimenſions n, where A is ſuppoſed to repreſent the abſolute known Term of the Equation. Let k repreſent the Limit next leſs than any of the Roots, and ſuppoſing $x = k + f$, ſubſtitute the Powers of $k+f$ inſtead of the Powers of x, and there will ariſe, $\overline{k+f}^n + p \times \overline{k+f}^{n-1} + q \times \overline{k+f}^{n-2} + r \times \overline{k+f}^{n-3}$, &c. $+ A = 0$, or by Involution, diſpoſing the Terms according to the Dimenſions of f . . . .

$$\left.\begin{array}{l}
k^n + n k^{n-1} \times f + n \times \frac{n-1}{2} k^{n-2} f^2 +, \text{\&c.} \\
p k^{n-1} + p \times \overline{n-1}\, k^{n-2} \times f + p \times \overline{n-1} \times \frac{n-2}{2} k^{n-3} f^2 +, \text{\&c.} \\
q k^{n-2} + q \times \overline{n-2}\, k^{n-3} \times f + q \times \overline{n-2} \times \frac{n-3}{2} k^{n-4} f^2 +, \text{\&c.} \\
r k^{n-3} + r \times \overline{n-3}\, k^{n-4} \times f + r \times \overline{n-3} \times \frac{n-4}{2} k^{n-5} f^2 +, \text{\&c.} \\
\text{\&c.}
\end{array}\right\} + A = 0,$$

where neglecting all the Powers of f after the firſt two Term
you find

$$f = \frac{-A - k^n - pk^{n-1} - qk^{n-2} - rk^{n-3},\ \&c.}{nk^{n-1} + p \times \overline{n-1}k^{n-2} + q \times \overline{n-2}k^{n-3} + r \times \overline{n-3}k^{n-4},\ \&}$$

Then $k + f = x$, the Root, which by this Series may be had
any required Exactneſs in any Equation of the Power n.

573. " By this Method you may diſcover Theorems for ap
proximating to the Roots of *pure Powers*;" as to find the n Ro
of any Number A; ſuppoſe k to be the neareſt leſs Root in In
tegers, and that $k + f$ is the true Root, then ſhall $k^n + nk^{n-}$
$f + n \times \frac{n-1}{2} k^{n-2} f^2$, &c. $= A$; and aſſuming only th
two firſt Terms, $f = \frac{A - k^n}{nk^{n-1}}$: Or, more nearly, taking th
three firſt Terms,

$$f = \frac{A - k^n}{nk^{n-1} + n \times \frac{n-1}{2} k^{n-2} f}, \text{ and (taking } \frac{A - k^n}{nk^{n-1}} =$$

$$f = \frac{A - k^n}{nk^{n-1} + \frac{nn-n}{2} k^{n-2} \times \frac{A - k^n}{nk^{n-1}}} = \frac{A - k^n}{nk^{n-1} + \frac{n-n}{2k} \times \overline{A-}}$$

$$= (\text{putting } m = A - k^n). = \frac{km}{nk^n + \frac{n-1}{2} \times m}; \text{ which is}$$

rational Theorem for approximating to f.

574. You may find an *irrational* Theorem for it by aſſum
ing the three firſt Terms of the Power of $k + f$, *viz.* k^n +
$nk^{n-1}f + n \times \frac{n-1}{2} k^{n-2} f^2 = A$.

For, $nk^{n-1}f + n \times \frac{n-1}{2} k^{n-} f^2 = A - k^n = m$; an
reſolving this Quadratic Equation, you find

$$f = -\frac{k}{n-1} \pm \sqrt{\frac{2m}{n \times \overline{n-1} \times k^{n-2}} + \frac{k^2}{\overline{n-1}|^2}} =$$

$$= -\frac{k}{n-1} \pm \sqrt{\frac{2mn - 2m + nk^n}{n \times \overline{n-1}|^2 \, k^{n-2}}}.$$

575. In the Application of theſe Theorems, when a near alue of f is obtained, then adding it to k ſubſtitute the Aggrete in Place of k in the Formula, and you will, by a new Opetion, obtain a more correct Value of the Root required; and, thus proceeding, you may arrive at any Degree of Exactſs.

Thus, to obtain the Cube Root of 2, ſuppoſe $k = 1$, and $(= \frac{km}{nk^n + \frac{n-1}{2}m}) = \frac{1}{4} = 0.25$. In the ſecond Place, ppoſe $k = 1.25$, and f will be found by a new Operation, ual to 0.009921, and conſequently, $\sqrt[3]{2} = 1.259921$ near-. By the irrational Theorem, the ſame Value is diſcovered r $\sqrt[3]{2}$.

CHAP. XXIX.

f the Method of Series by which you may approximate to the Roots of Literal Equations.

6. IF there be only two Letters, x and a, in the propoſed Equation, ſuppoſe a equal to Unit, and find the Root the numeral Equation that ariſes from the Subſtitution, by e Rules of the laſt Chapter. Multiply theſe Roots by a, d the Products will give the Roots of the propoſed Equation.

Thus the Roots of the Equation $x^2 - 16x + 55 = 0$ e found to be 5 and 11. And therefore the Roots of the quation $x^2 - 16ax + 55a^2 = 0$, will be $5a$ and $11a$. he Roots of the Equation $x^3 + a^2x - 2a^3 = 0$ are found

by

by enquiring what are the Roots of the numeral Equation $x^3 + x - 2 = 0$, and ſince one of theſe is 1, it follows, that one of the Roots of the propoſed Equation is *a*; the other two are *imaginary.*

577. If the Equation to be reſolved involves more than two Letters, as $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$, then the Value of *x* may be exhibited in a Series having its Terms compoſed of the Powers of *a* and *y* with their reſpective Coefficients; which will *converge the ſooner the leſs* y *is in reſpect of* a, *if the Terms are continually multiplied by the Powers of* y, *and divided by thoſe of* a. Or, *will converge the ſooner the greater* y *is in reſpect of* a, *if the Terms be continually multiplied by the Powers of* a, *and divided by thoſe of* y. Since when *y* is very little in reſpect of *a*, the Terms $y, \frac{y^2}{a}, \frac{y^3}{a^2}, \frac{y^4}{a^3}, \frac{y^5}{a^4}$, &c. decreaſe very quickly. If *y* vaniſh in reſpect of *a*, the ſecond Term will vaniſh in reſpect of the firſt, ſince $\frac{y^2}{a} : y :: y : a$. And after the ſame Manner $\frac{y^3}{a^2}$ vaniſhes in reſpect of the Term immediately preceding it.

578. But when *y* is vaſtly great in reſpect of *a*, than *a* is vaſtly great in reſpect of $\frac{a^2}{y}$, and $\frac{a^2}{y}$ in reſpect of $\frac{a^3}{y^2}$; ſo that the Terms $a, \frac{a^2}{y}, \frac{a^3}{y^2}, \frac{a^4}{y^3}, \frac{a^5}{y^4}$, &c. in this Caſe *decreaſe* very ſwiftly. In either Caſe, the Series converge ſwiftly that conſiſt of ſuch Terms; and a few of the firſt Terms will give a near Value of the Root required.

EXAMPLE *of Caſe* I.

579. If a Series for *x* is required from the propoſed Equation that ſhall converge the ſooner the leſs *y* is in reſpect of *a*; to find the firſt Term of this Series, we ſhall ſuppoſe *y* to vaniſh; and extracting the Root of the Equation $x^3 + a^2x - 2a^3 = 0$ conſiſting of the remaining Parts of the Equation that do not vaniſh with *y*, we find, (by Inſt. 576.) thst $x = a$; which is the true Value of *x* when *y* vaniſhes, but is only near its Value when *y* does not vaniſh, but only is very little. To get a Value ſtill nearly the true Value of *x*, ſuppoſe the Difference of *a* from

the true Value to be p, or that $x = a + p$. And ſubſtituting $a + p$ in the given Equation for x, you will find,

$$\left.\begin{array}{r}
x^3 = a^3 + 3a^2p + 3ap^2 + p^3 \\
+ a^2x = a^3 + a^2p \\
- 2a^3 = - 2a^3 \\
+ ayx = a^2y + apy \\
- y^3 = - y^3
\end{array}\right\} = 0; \text{ or}$$

$$x^3 + a^2x - 2a^3 + ayx - y^3 = \left.\begin{array}{r} 4a^2p + 3ap^2 + p^3 \\ a^2y + apy - y^3 \end{array}\right\} = 0.$$

But ſince, by Suppoſition, y and p are very little in reſpect of a, it follows that the Terms $4a^2p$, 2^2y, where y and p are ſeparately of the *leaſt* Dimenſions, are vaſtly great in reſpect of the reſt; ſo that, in determining a near Value of p, the reſt may be neglected: And from $4a^2p + a^2y = 0$, we find $p = -\frac{1}{4}y$. So that $x = a + p = a - \frac{1}{4}y$, nearly.

Then to find a nearer Value of p, and conſequently of x, ſuppoſe $p = -\frac{1}{4}y + q$, and ſubſtituting this Value for it in the laſt Equation, you will find,

$$\left.\begin{array}{rl}
p^3 = & -\frac{1}{64}y^3 + \frac{3}{16}y^2q - \frac{3}{4}yq^2 + q^3 \\
3ap^2 = & \frac{3}{16}ay^2 - \frac{3}{2}ayq + 3aq^2 \\
4a^2p = & - a^2y + 4a^2q \\
ayp = & -\frac{1}{4}ay^2 + ayq \\
a^2y = & a^2y \\
-y^3 = & -y^3
\end{array}\right\} = 0 =$$

$$= \left.\begin{array}{l}
-\frac{65}{64}y^3 + \frac{3}{16}y^2q - \frac{3}{4}yq^2 + q^3 \\
-\frac{1}{16}ay^2 - \frac{1}{2}ayq + 3aq^2 \\
+ 4a^2q
\end{array}\right\} = 0.$$

And ſince, by the Suppoſition, q is very little in reſpect of p, which is nearly $= -\frac{1}{4}y$, therefore q will be very little in reſpect of y; and conſequently all the Terms of the laſt Equation will be very little in reſpect of theſe two, *viz.* $-\frac{1}{16}ay^2$, $+ 4a^2q$, where y and q are of leaſt Dimenſions ſeparately: Particularly the Term $-\frac{1}{2}ayq$ is little in reſpect of $4a^2q$, becauſe y is very little in reſpect of a; and it is little in reſpect of $-\frac{1}{16}ay^2$ becauſe q is little in reſpect of y.

Neglect therefore the other Terms, and ſuppoſing $-\frac{1}{16}ay^2$

$+ 4a^2 q = 0$, you will have $q = \frac{1}{64} \times \frac{y^2}{a}$; ſo that $x = a -$ $\frac{1}{4}y + \frac{1}{64} \times \frac{y^2}{a}$. And by proceeding in the ſame Manner you will find $x = a - \frac{y}{4} + \frac{y^2}{64a} - \frac{131y^3}{512a^2} + \frac{509y^4}{16384a^3} -$, &c.

EXAMPLE *of Caſe* II.

580. When it is required to find a Series for x that ſhall converge ſooner, the *greater* y is in reſpect of any Quantity a, you need only ſuppoſe a to be very little in reſpect of y, and proceed by the ſame Reaſoning as in the laſt Example on the Suppoſition of y being very little.

Thus, to find a Value for x in the Equation $x^3 - a^2x + ayx - y^3 = 0$ that ſhall converge the ſooner the greater y is in reſpect of a. Suppoſe a to vaniſh, and the remaining Terms will give $x^3 - y^3 = 0$, or $x = y$. So that when y is vaſtly great, it appears that $x = y$ nearly.

But to have the Value of x more accurately, put $x = y + p$, then,

$$\left.\begin{aligned} x^3 &= y^3 + 3y^2p + 3yp^2 + p^3 \\ -a^2x &= -a^2y - a^2p \\ +ayx &= ay^2 + ayp \\ -y^3 &= -y^3 \end{aligned}\right\} = 0 =$$

$$\begin{aligned} = &+ 3y^2p + 3yp^2 + p^3 - a^2y - a^2p \\ &+ ay^2 + ayp: \end{aligned}$$

Where the Terms $3y^2p + ay^2$ become vaſtly greater than the reſt, y being vaſtly greater than a or p; and conſequently $p = -\frac{1}{3}a$ nearly.

Again, by ſuppoſing $p = -\frac{1}{3}a + q$ you will transform the laſt Equation into

$$\left.\begin{aligned} &-\tfrac{3}{27}a^3 + 3y^2q + 3yq^2 + q^3 \\ &-a^2y - ayq - aq^2 \\ &-\tfrac{2}{3}a^2q \end{aligned}\right\} = 0.$$

Where the two Terms $3qy^2 - a^2y$ muſt be vaſtly greater than any of the reſt, a being vaſtly leſs than y, and q vaſtly leſs than by the Suppoſition; ſo that $3qy^2 - a^2y = 0$, and $q = \frac{a^2}{3y}$ nearly. By proceeding in this Manner, you may correct the Value of y and find that

$$x = y - \tfrac{1}{3}a + \frac{a^2}{3y} + \frac{a^3}{81y^2} - \frac{8a^4}{243y^3}, \text{ \&c.}$$

Which Series converges the ſooner the greater y is ſuppoſed to be taken in reſpect of a.

581. In the Solution of the firſt Example thoſe Terms were always compared in order to determine p, q, r, &c. in which y and thoſe Quantities p, q, r, &c. were *ſeparately* of feweſt Dimenſions. But in the ſecond Example, thoſe Terms were compared in which a and the Quantities p, q, r, &c. were of leaſt Dimenſions ſeparately. And theſe always are the proper Terms to be compared together, becauſe they become vaſtly greater than the reſt, in the reſpective Hypotheſes.

582. *In general*; to determine the firſt, or any, Term in the Series, *ſuch Terms of the Equation are to be aſſumed together only, as will be found to become vaſtly greater than the other Terms*; that is, which give a Value of x which ſubſtituted for it in all the Terms of the Equation ſhall raiſe the Dimenſions of the other Terms all above, or all below, the Dimenſions of the aſſumed Terms, according as y is ſuppoſed to be vaſtly little, or vaſtly great in reſpect of a.

Thus to determine the firſt Term of a converging Series expreſſing the Value of x in the laſt Equation $x^3 - a^2x + ayx - y^3 = 0$, the Terms ayx and $-y^3$ are not to be compared together, for they would give $x = \frac{y^2}{a}$, which ſubſtituted for x the Equation becomes,

$\frac{y^6}{a^3} - ay^2 + y^3 - y^3 = 0$, where the firſt Term is of more Dimenſions than the aſſumed Terms ayx, $-y^3$: and the 2d of fewer; ſo that the two firſt Terms cannot be neglected in reſpect of the two laſt, neither when y is very great nor very little, compared with a. Nor are the Terms x^3, ayx, fit to be compared together in order to obtain the firſt Term of a Series for x, for the like Reaſon.

But x^3 may be compared with $-a^2x$, as alſo $-a^2x$ wit $-y^3$, for that End. Theſe two give the firſt Terms of a Series that converges the ſooner the leſs y is; as $x^3 = y^3$ gives th firſt Term of a Series that converges the ſooner the greater y i The laſt Series was given in the preceding Article. The comparing x^3 with $-a^2x$ gives theſe two Series,

$$x = a - \tfrac{1}{2}y - \frac{y^2}{8a} + \frac{7y^3}{16a^2} - \frac{59y^4}{128a^3}, \&c.$$

$$x = -a + \tfrac{1}{2}y + \frac{y^2}{8a} + \frac{9y^3}{16a^2} + \frac{69y^4}{128a^3}, \&c.$$

The comparing $-a^2x$ with $-y^3$ gives,

$$x = -\frac{y^3}{a^2} - \frac{y^4}{a^3} - \frac{y^5}{a^4} - \frac{y^6}{a^5} -, \&c.$$

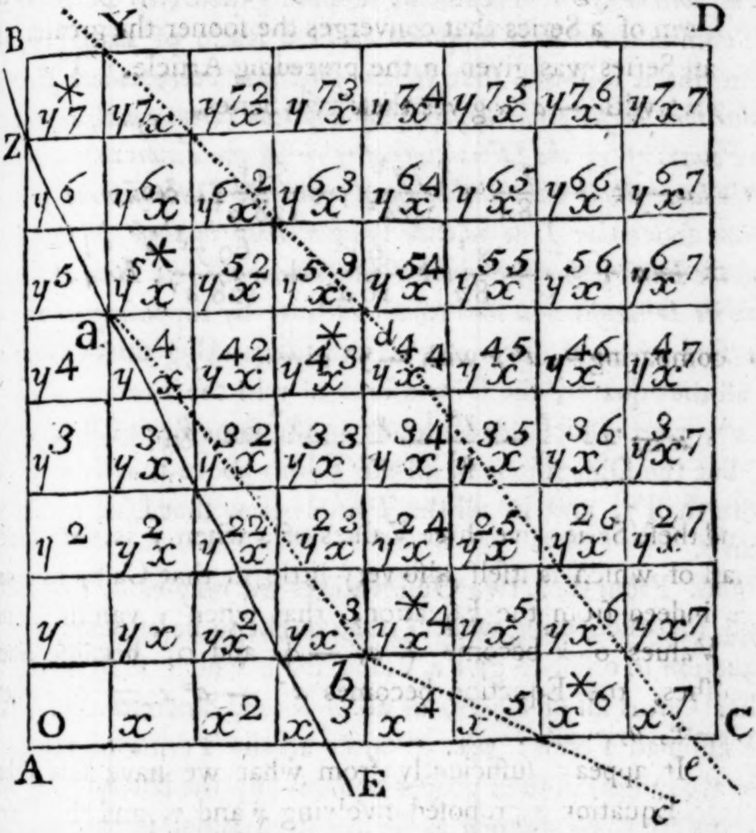
And theſe Series give three Values of x when y is very little the laſt of which is itſelf alſo very little in that Caſe, as it appears indeed from the Equation, that when y vaniſhes, th three Values of x become $+a$, $-a$, and 0, becauſe whe y vaniſhes, the Equation becomes $x^3 - a^2x = 0$, whoſ Roots are, a, $-a$, 0.

583. It appears ſufficiently from what we have ſaid, tha when an Equation is propoſed involving x and y, and the Valu of x is required in a converging Series, the Difficulty of finding the firſt Term of the Series is reduced to this; *to find what Term aſſumed in order to determine a Value of* x *expreſſed in ſome Dimenſion of* y *and* a *will give ſuch a Value of it as ſubſtituted for it in the othe Terms will make them all of more Dimenſions of* y, *or all of leſs Dimenſions of* y, *than thoſe aſſumed Terms.*

To determine this, draw B A and A C at right Angles to each other, compleat the Parallelogram A B C D and divide it into equal Squares, as in the Figure. In theſe Squares place th Powers of x from A towards C, and the Powers of y from A towards B, and in any other Square place that Power o x that is directly below it in the Line A C, and that Power o y that is in a Parallel with it in the Line A B; ſo that the Index of x in any Square may expreſs its Diſtance from the Line A B

an

nd the Index of y in any Square may expreſs its Diſtance from he Line A C. Of this Square we are to obſerve,



$\overset{*}{y^7}$	y^7x	y^7x^2	y^7x^3	y^7x^4	y^7x^5	y^7x^6	y^7x^7
y^6	y^6x	y^6x^2	y^6x^3	y^6x^4	y^6x^5	y^6x^6	y^6x^7
y^5	$\overset{*}{y^5x}$	y^5x^2	y^5x^3	y^5x^4	y^5x^5	y^5x^6	y^6x^7
y^4	y^4x	y^4x^2	$\overset{*}{y^4x^3}$	y^4x^4	y^4x^5	y^4x^6	y^4x^7
y^3	y^3x	y^3x^2	y^3x^3	y^3x^4	y^3x^5	y^3x^6	y^3x^7
y^2	y^2x	y^2x^2	y^2x^3	y^2x^4	y^2x^5	y^2x^6	y^2x^7
y	yx	yx^2	yx^3	$\overset{*}{yx^4}$	yx^5	yx^6	yx^7
0	x	x^2	x^3	x^4	x^5	$\overset{*}{x^6}$	x^7

584. 1. That the Terms are not only in Geometrical Progreſ-ion in the Vertical Column A B, or the Horizontal A C, and their arallels; but alſo in the Terms taken in any oblique ſtrait Line hatever; for in any ſuch Terms it is manifeſt that the *Indices* f y and x will be in arithmetical Progreſſion. The Indices of , becauſe thoſe Terms will remove equally from the Line A C, r approach equally to it, and the Indices of y in any ſuch Terms re as their Diſtances from that Line A C. The Indices of x ill alſo be in arithmetical Progreſſion, becauſe theſe Terms qually remove from, or approach to the Line A B. Thus for xample, in the Terms y^7, y^5x, y^3x^2, yx^3, the Indices of y ecreaſing by the common Difference 2, while the Indices of x ncreaſe in the Progreſſion of the natural Numbers, the common atio of the Terms is $\frac{x}{y^2}$. It follows,

585. 2.

585. 2. From the laſt Obſervation, that *if any two Terms be*
poſed equal, then all the Terms in the ſame ſtrait Line with
Terms will be equal: becauſe by ſuppoſing theſe two Terms eq
the common Ratio is ſuppoſed to be a Ratio of Equality;
from this it follows, that *if you ſubſtitute every where for*
Value that ariſes for it by ſuppoſing any two Terms equal, expreſſ
the Powers of y, *the Dimenſions of* y *in all the Terms that are f*
in the ſame ſtrait Line will be equal; but *the Dimenſions of* y *in*
Terms above *that Line will be* greater *than in thoſe in that L*
and *the Dimenſions of* y *in the Terms* below *the ſaid Line will be*
than its Dimenſions in that Line. Thus, by ſuppoſing $y^7 =$
we find $x^3 = y^6$, or $x = y^2$; and ſubſtituting this Value f
in all the Squares, the Dimenſions of y in the Terms, y^7, y
$y^3 x^2$, $y x^3$, which are all found in the ſame ſtrait Line, wil
7, but the Dimenſions in all the Terms above that Line wil
more than 7, and in all the Terms below that Line will be
than 7.

586. From theſe two Obſervations we may eaſily find a
thod for diſcovering what Terms ought to be aſſumed from
Equation in order to give a Value for x which ſhall make
other Terms all of *higher*, or all of *lower* Dimenſions of y t
the aſſumed Terms: *viz.* " after all the Terms of the Eq
tion are ranged in their proper Squares (by the laſt Article) ſ
Terms are to be aſſumed as lie in a ſtrait Line, ſo that the o
Terms either lie all above the ſtrait Line, or fall all below i

For Example, ſuppoſe the Equation propoſed is $y^7 - a$
$+ y^4 x^3 + a^2 y x^4 - a x^6 = 0$, then marking with an A
riſk the Squares in the laſt Article which contain the ſame
menſions of x and y as the Terms in the Equation, imagi
Ruler ZE to revolve about the firſt Square at Z marked a
and as it moves from A towards C, it will firſt meet the T
$a y^5 x$, and while the Ruler joins theſe two Terms, all the o
Terms lie above it: from which you infer that by ſuppo
theſe Terms equal, (*viz.* $y^7 = a y^5 x$) you ſhall obtain a V
of x, which ſubſtituted for it, will give all the other Terms of hig
Dimenſions of y, than thoſe Terms: And hence we concl
that the Value of x deduced from ſuppoſing theſe Terms eq
viz. $\frac{y^2}{a}$, is the firſt Term of a Series that will converge
ſooner the leſs y is in reſpect of a.

587. If the Ruler be made to revolve about the ſame Square,
Y, the contrary Way from D towards C, it will firſt meet
e Term $y^4 x^3$, and by ſuppoſing $y^7 + y^4 x^3 = 0$, we find y
x, which gives the firſt Term of a Series for x, that con-
rges the ſooner the greater that y is. And this is the cele-
ated Rule invented by Sir *Iſaac Newton*, for this Purpoſe.

588. This Rule may be extended to Equations having Terms
at involve Powers of x and y with *Fractional* or *Surd* Indices;
by taking Diſtances from A in the Lines AC and AB pro-
rtional to theſe Fractions and Surds," and thence determining
e Situation of the Terms of the propoſed Equation in the Pa-
llelogram ABCD.

589. It is to be obſerved alſo, that when the Line joining
y two Terms has all the other Terms on one Side of it, by
em you may find the firſt Term of a converging Series for x,
d thus "various ſuch Series can be deduced from the ſame
quation." As, in the laſt Example, the Line ab joining $y^5 x$
d $y x^4$ has all the Terms above it; and therefore ſuppoſing
$-ay^5 x + a^2 y x^4 = 0$ we find $x^3 = \frac{y^4}{a}$, and $x = \frac{y^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, which
the firſt Term of another converging Series for x. Again,
e ſtrait Line bc joining $y x^4$ and x^6 has all the other Terms a-
ve it, and therefore, ſuppoſing $a^2 y x^4 - a x^6 = 0$, we find
$= x^2$, and $x = a^{\frac{1}{2}} y^{\frac{1}{2}}$, the firſt Term of another Series for
, converging alſo the ſooner the leſs y is. There are two Se-
es converging the ſooner the greater y is, to be deduced from
ppoſing $y^7 = -y^4 x^3$, by the Line Yd, or $y^4 x^3 = a x^6$, by
e Line de. And, to find all theſe Series, "deſcribe a *Poly-*
n Zabed, having a Term of the Equation in each of its An-
es, and including all the other Terms within it, then a Series
ay be found for x, by ſuppoſing any two Terms equal that are
aced in any two adjacent Angles of the Polygon."

590. If the Ruler ZE be made to move parallel to itſelf, all
e Terms which it will touch at once will be of the ſame Di-
enſions of y: for they will bear the ſame Proportion to one
other as the Terms in the Line ZE themſelves. The Terms
hich the Ruler will touch firſt will have fewer Dimenſions of
than thoſe it touches afterwards in the Progreſs of its Motion,
it moves towards D; but more Dimenſions than they, if it

moves towards A. The Terms in the ſtrait Line ZE, ſerve to determine the firſt Term of the converging Series required.

Theſe with the Terms it touches afterwards ſerve to determine the ſucceeding Terms of the converging Series; all the reſt vaniſhing compared with theſe, when *y* is very little and the Ruler moves from A towards D, or when *y* is vaſtly great and the Ruler moves from D towards A.

591. The ſame Author* gives another Method for diſcovering the firſt Term of a Series that ſhall converge the ſooner the leſs *y* is. “Suppoſe the Term where *y* is ſeparately of feweſt Dimenſions to be Dy^l; compare it ſucceſſively with the other Terms, as with $Ey^m x^s$, and obſerve where $\frac{l-m}{s}$ is found *greateſt*; and putting $\frac{l-m}{s}=n$, Ay^n will be the firſt Term of a Series that ſhall converge the ſooner the leſs *y* is:” For in that Caſe Dy^l and $Ey^m x^s$ will be infinitely greater than any other Terms of the propoſed Equation. Suppoſe $Fy^e x^k$ is any other Term of the Equation, and, by the Suppoſition, $\frac{l-m}{s}$ $(=n)$ is greater than $\frac{l-e}{k}$, and conſequently, multiplying by *k*, you find nk greater than $l-e$, and $nk+e$ greater than l; now if for *x* you ſubſtitute Ay^n, then $Fy^e x^k = FA^k y^{nk+e}$, which therefore will vaniſh compared with Dy^l (ſince $nk+e$ is greater than l) when *y* is infinitely little. Thus therefore all the Terms will vaniſh compared with Dy^l and $Ey^m x^s$ which are ſuppoſed equal; and conſequently they will give the firſt Term of a Series that will converge the ſooner the leſs *y* is.

592. If you obſerve “when $\frac{l-m}{s}$ is found *leaſt* of all, and ſuppoſe it equal to *n*, then will Ay^n be the firſt Term of a Series that will converge the ſooner the *greater* *y* is.” For in that Caſe Dy^l and $Ey^m x^s$ will be infinitely greater than $Fy^e x^k$, becauſe $\frac{l-m}{s}$ $(=n)$ being leſs than $\frac{l-e}{k}$, it follows that nk is leſs than $l-e$, and $nk+e$ leſs than l, and conſequently $Fy^e x^k$ $(=FA^k y^{nk+e})$ vaſtly leſs than Dy^l, when *y* is very great.

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* Sir *I. Newton*.

593. After the ſame Manner, if you compare any Term $Dy^l x^b$, where both x and y are found, with all the other Terms, and obſerve where $\frac{l-m}{s-b}$ is found the *greateſt* or *leaſt*, and ſuppoſe $\frac{l-m}{s-b}=n$, then may Ay^n be the firſt Term of a converging Series. For ſuppoſing that $Fy^e x^k$ is any other Term of the Equation, if $\frac{l-m}{s-b}(=n)$ is greater than $\frac{l-e}{k-b}$, then ſhall $nk-nb$ be greater than $l-e$, and $nk+e$ greater than $l+nb$. But $nk+e$ are the Dimenſions of y in $Fy^e x^k$ when $x=Ay^n$, and $l+nb$ are the Dimenſions of y in $Ey^m x^s$; therefore $Fy^e x^k$ is of more Dimenſions of y than $Ey^m x^s$, and therefore vaniſhes compared to it when y is ſuppoſed infinitely little. In the ſame Manner, if $\frac{l-m}{s-b}$ is leſs than $\frac{l-e}{k-b}$, then will $Ey^m x^s$ be infinitely greater than $Fy^e x^k$, when y is infinite.

594. When the firſt Term (Ay^n) of the Series is found by the preceding Method, then by ſuppoſing $x = Ay^n + p$, and ſubſtituting this Binomial and its Powers for x and its Powers, there will ariſe an Equation for determining p the ſecond Term of the Series. This new Equation may be treated in the ſame Manner as the Equation of x, and by the Rule of Inſt. 586, the Terms that are to be compared in order to obtain a near Value of p, may be diſcovered; by Means of which Terms p may be found: Which ſuppoſe equal to By^{n+r}, then by ſuppoſing $p = By^{n+r} + q$, the Equation may be transformed into one for determining q the third Term of the Series, and by proceeding in the ſame Manner you may determine as many Terms of the Series as you pleaſe; finding $x = Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, *&c.* where the Dimenſions of y aſcend or deſcend according as r is poſitive or negative; and always "in Arithmetical Progreſſion, that this Value of x being ſubſtituted for it in the propoſed Equation, the Terms involving y and its Powers may fall in with one another, ſo that more than one may always involve the ſame Dimenſion of y, which may mutually deſtroy each other and make the whole Equation vaniſh, as it ought to do.".

595. It is obvious that as the Dimenſions of y in Ay^n + $By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. are in an Arithmetical Progreſſion whoſe Difference is r, the Square, Cube, or any Power s of $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. will conſiſt of Terms wherein the Dimenſions of y will conſtitute an Arithmetical Progreſſion having the ſame common Difference r; for theſe Dimenſions will be $sn, sn + r, sn + 2r, sn + 3r$, &c. Therefore, if in any Term $Ey^m x^s$ you ſubſtitute for x the Series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$ &c. the Terms of the Series expreſſing $Ey^m x^s$ will conſiſt of theſe Dimenſions of y, viz. $m + sn, m + sn + r, m + sn + 2r, m + sn + 3r$ &c. and by a like Subſtitution in any other Term as $Fy^e x^k$, the Dimenſions of y will be $e + nk, e + nk + r, e + nk + 2r, e + nk + 3r$ &c. The former Series of Indices muſt coincide with the latter Series, that the Terms in which they are found may be compared together, and be found equal with oppoſite Signs ſo as to deſtroy one another, and make the whole Equation vaniſh.

596. The firſt Series conſiſts of Terms ariſing by adding ſome Multiple of r to $m + sn$, the latter by adding ſome Multiple of r to $e + nk$; and that theſe may coincide, ſome Multiple of r added to $m + sn$ muſt be equal to ſome other Multiple of r added to $e + nk$. From which it appears, that the Difference of $m + sn$ and $e + nk$ is always a Multiple of r; and conſequently that r is a Diviſor of the Difference of Dimenſions of y in the Terms $Ey^m x^s$ and $Fy^e x^k$, ſuppoſing $x = Ay$ It follows therefore "that r is a common Diviſor of the Differences of the Dimenſions of y in the Terms of the Equation when you have ſubſtituted Ay^n for x in all the Terms." And i r be aſſumed equal to the *greateſt* common Diviſor (excepting ſome Caſes afterward to be mentioned) you will have the tru Form of a Series for x. And now the Dimenſions y^n, y^{n+r}, y^{n+2r}, y^{n+3r} &c. being known, there remains only, by Calculation to determine the general Coefficients A, B, C, D, &c in order to find the Series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. $= x$.

597. This leads us to Sir *Iſaac Newton*'s *Second General Metho* of Series; which conſiſts in aſſuming a Series with undetermined Coefficients expreſſing x, as $Ay^n + By^{n+r} + Cy^{n+2r} +$

&c

&c. Where A, B, C, &c. are ſuppoſed as yet unknown, but and r are diſcovered by what we have already demonſtrated; nd ſubſtituting this every where for x, you muſt ſuppoſe, in he new Equation that ariſes, the Sum of all the Terms that involve the ſame Dimenſion of y to vaniſh, by which Means you will obtain particular Equations, the *firſt* of which will give A, the *ſecond* B, the *third* C, &c. and theſe Values being ſubſtituted in the aſſumed Series for A, B, C, &c. the Series for x will be obtained as far as you pleaſe.

598. Let us apply, for Example, this Method to the Equation (of Inſt. 577.) $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$. Suppoſe it is required to find a Series converging the ſooner the leſs y is: Its firſt Term (by Inſt. 579, or 583) is found to be a, ſo that $n = 0$. Subſtitute a for x in the Equation, and the Terms become $a^3 + a^3 - 2a^3 + a^2y - y^3$, and the Differences of the Indices are 0, 1, 2, 3; whoſe greateſt common Meaſure is 1, ſo that $r = 1$. Aſſume therefore $x = A + By + Cy^2 + Dy^3$ &c. and ſubſtitute this Series for x in the Equation. Then

$$\begin{aligned}
x^3 &= A^3 + 3A^2By + 3AB^2y^2 + B^3y^3 + \&c. \\
&\qquad\qquad + 3A^2Cy^2 + 3A^2Dy^3 + \&c. \\
&\qquad\qquad\qquad + 6ABCy^3 + \&c. \\
+a^2x &= a^2A + a^2By + a^2Cy^2 + a^2Dy^3 + \&c. \\
+ayx &= \qquad aAy + aBy^2 + aDy^3 + \&c. \\
-2a^3 &= -2a^3 \\
-y^3 &= \ldots\ldots\ldots\ldots -1 \times y^3.
\end{aligned}$$

599. Now ſince $x^3 + a^2x + ayx - 2a^3 - y^3 = 0$, it follows that the Sum of theſe Series involving y muſt vaniſh. But that cannot be if the Coefficient of every particular Term does not vaniſh. For every Term where y is infinitely little, is infinitely greater than the following Terms, ſo that if every Term does not vaniſh of itſelf, the Addition or Subtraction of the following Terms which are infinitely leſs than it, or of the preceding Terms which are infinitely greater, cannot deſtroy it; and therefore the whole cannot vaniſh, It appears therefore that $A^3 + a^2A - 2a^3 = 0$, is an Equation for determining A, and gives $A = a$.

In order to determine B, you muſt ſuppoſe the Sum of the Coefficients affecting y to vaniſh, *viz.* $\overline{3A^2B + a^2B + aA} \times y = 0$, or ſince $A = a$, $4a^2By + a^2y = 0$, and $B = -\frac{1}{4}$.

To determine C, in the ſame Manner ſuppoſe $3AB^2y^2 +$ $A^2Cy^2 + a^2Cy^2 + aBy^2 = 0$, or ſubſtituting for A and their Values already found, $\frac{3ay^2}{16} + 4a^2Cy^2 - \frac{ay^2}{4} = 0$, a consequently $C = \frac{1}{64a}$. And, by proceeding in the ſam Manner, $D = \frac{131}{512a^2}$, ſo that $x = a - \frac{1}{4}y + \frac{1}{64a}y^2$ $\frac{131}{512a^2}y^3$, &c. as we found before in Inſt. 579.

600. By this Method you may transfer Series from one undetermined Quantity to another, and obtain Theorems for t *Reverſion* of Series.

Suppoſe that $x = ay + by^2 + cy^3 + dy^4 +$ &c. and it required to expreſs y by a Series conſiſting of the Powers of It is obvious that when x is very little, y is alſo very little, a that in order to determine the firſt Term of the Series, you ne only aſſume $x = ay$. And therefore $y = \frac{x}{a}$; ſo that $n = 1$. ſubſtituting $\frac{x}{a}$ for y, you find the Dimenſions of x in the Ter will be 1, 2, 3, 4, *&c.* ſo that $r = 1$ alſo. You may therefore aſſume $y = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. And the Subſtitution of this Value of y you will find,

$$\begin{array}{llll} ay = & aAx + & aBx^2 + & aCx^3 + \text{&c.} \\ by^2 = & & bA^2x^2 + & 2bABx^3 + \text{&c.} \\ cy^3 = & & & cA^3x^3 + \text{&c.} \\ \text{&c.} & & & \text{&c.} \end{array}$$

But the firſt Term being already found to be $\frac{x}{a}$, you have $= \frac{1}{a}$; an[illegible]ince $aB + bA^2 = 0$, it follows that $B = -$ After the ſame Manner you will find $C = \frac{2b^2 - ac}{a^4}$. Whenc

$$y = \frac{x}{a} - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^4}x^3 + \text{&c.}$$

601. Suppo

601. Suppoſe again you have $ax + bx^2 + cx^3 + dx^4 +$
c. $= gy + hy^2 + iy^3 + ky^4$ &c. to find x in Terms of y.
ou will eaſily ſee, (by Inſt. 586.) that the firſt Term of the Se-
es for x is $\frac{gy}{a}$, that $n = 1$, $r = 1$. Therefore aſſume $x =$
$y + By^2 + Cy^3$ &c. and by ſubſtituting this Value for x
d bringing all the Terms to one Side, you will have

$$\begin{array}{llll}
ax = aAy & + aBy^2 & + aCy^3 & + \&c. \\
bx^2 = & bA^2y^2 & + 2bABy^3 & + \&c. \\
cx^3 = & & cA^3y^3 & + \&c. \\
\&c. & & \&c. & \\
 & & & \\
-gy = -gy & & & \\
-hy^2 = . & . & . -hy^2 & \\
-iy^3 = . & . & . . . -iy^3 & \\
\&c. & & \&c. &
\end{array}$$

From whence we ſee, firſt, that $aA = g$, and $A = \frac{g}{a}$.

°. That $aB + bA^2 - h = 0$, and $B = \frac{h}{a} - \frac{bg^2}{a^3}$. 3°. That

$C + 2bAB + cA^3 - i = 0$, and therefore $C = \frac{i - 2bAB - cA^3}{a}$.

nd thus the three firſt Terms of the Series $Ay + By^2 + Cy^3$
c. are known.*

This Method is very general; but in ſome particular Caſes a
ifficulty will ariſe, which Mr. *Maclaurin* has ſhewn how to
void, to whoſe Treatiſe of Algebra we muſt refer the Reader,
s it would be very tedious here to dwell on that Subject.

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* See Mr. *De Moivre* in *Phil. Tranſ.* 240.

CHAP. XXX.

Mr. De Moivre's *Theorem for raiſing an Infin Series to any given Power. Alſo the Evoluti or Reverſion of Series deduced from thence.*

602. IN the Philoſophical Tranſactions,* the late Mr. *Moivre* has given us the following excellent Theor for raiſing any Multinomial or infinite Series to any gi Power, *viz.*

$$\overline{az+bz^2+cz^3+dz^4+ez^5+fz^6,\ \&c.}^{\,m}\ \text{will be}$$

$$= a^m z^m$$

$$+\frac{m}{1}a^{m-1}bz^{m+1}$$

$$\left.\begin{matrix}+\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^2\\ +\frac{m}{1}a^{m-1}c\end{matrix}\right\}z^{m+2}$$

$$\left.\begin{matrix}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3\\ +\frac{m}{1}\times\frac{m-1}{1}a^{m-2}bc\\ +\frac{m}{1}a^{m-1}d\end{matrix}\right\}z^{m+3}$$

$$\left.\begin{matrix}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}a^{m-4}b^4\\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^2c\\ +\frac{m}{1}\times\frac{m-1}{1}a^{m-2}bd\\ +\frac{m}{1}\times\frac{m-1}{2}a^{m-2}c^2\\ +\frac{m}{1}a^{m-1}e\end{matrix}\right\}z^{m+4}$$

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* N°. 240.

$$\left.\begin{array}{r}
\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}a^{m-5}b^{5}\\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}a^{m-4}b^{3}c\\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^{2}d\\
+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{2}a^{m-3}bc^{2}\\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-2}be\\
\frac{m}{1}+\frac{m-1}{1}a^{m-2}cd\\
\times\frac{m}{1}a^{m-1}f
\end{array}\right\}z^{m+5}$$

$$\left.\begin{array}{r}
\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}a^{m-6}b^{6}\\
\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{1}a^{m-5}b^{4}c\\
\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}a^{m-4}b^{3}d\\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\times\frac{m-3}{2}a^{m-4}b^{2}c^{2}\\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^{2}e\\
+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{1}a^{m-3}bcd\\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-3}bf\\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}c^{3}\\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-2}ce\\
+\frac{m}{1}\times\frac{m-1}{2}a^{m-2}d^{2}\\
\frac{m}{1}a^{m-1}g
\end{array}\right\}z^{m+6},\ \&c.$$

603. For underſtanding of which, it is only neceſſary to c
ſider all the Terms by which the ſame Power of z is multipl
In order to which two things in each of theſe Terms mu
conſidered, 1°, The Product of certain Powers of the g
Quantities of Coefficients a, b, c, d, &c. And, 2°, The
ciæ or Products of $\frac{m}{1} \times \frac{m-1}{2}$, &c. prefixed to them.

Now to find all the Products belonging to the ſame Pow
z. For Example; to find that Product whoſe Index is m
(r being any whole Number) the ſaid Products muſt be di
guiſhed into ſeveral Claſſes. Thoſe which immediately
ſome certain Power of a (by which all theſe Products begin)
Products of the firſt Claſs: As $a^{m-4} b^5 e$ is a Product of the
Claſs, becauſe b immediately follows a^{m-4}. Thoſe which
mediately after ſome Power of a have c, are Products of th
cond Claſs. So $a^{m-1} ccd$ is a Product of the ſecond C
Thoſe which immediately after ſome Power of a have d, are
ducts of the third Claſs, and ſo of the reſt.

This being underſtood, 1°, Multiply all the Products bel
ing to z^{m+r-1} (which immediately precedes z^{m+r}) b
and divide them all by a. 2°, Multiply by c, and di
by a, all the Products belonging to z^{m+r-2} except tho
the firſt Claſs. 3°, Multiply by d, and divide by a all the
ducts belonging to z^{m+r-3} except thoſe of the firſt and ſe
Claſs. 4°, Multiply by e, and divide by a, all the Terms
longing to z^{m+r-4}, except thoſe of the firſt, ſecond, and t
Claſs; and ſo on, till you meet twice with the ſame T
Laſtly, add the Product of a^{m-r} into the Letter whoſe Expo
is $r - 1$ to all theſe Terms.

Note, The Exponent of a Letter is the Number expre
what Place that Letter has in the Alphabet, as is 3 the Expo
of the Letter c, being the 3d Letter.

By this Rule it is manifeſt that it is eaſy to find all the
ducts belonging to the ſeveral Powers of z, if you have but
Product belonging to z^m, viz. a^m.

604. Next to find the *Unciæ* prefixed to every Product,
muſt conſider the Sum of the Units contained in the Expon
of the Letters that compoſe it (the Index of a excepted); th
write as many Terms of the Series $m \times m - 1 \times m -$

$m -$

-3, &c. as there are Units in the Sum of theſe Indexes; Series is to be the Numerator of a Fraction, whoſe Denomi r is the Product of the ſeveral Series 1 × 2 × 3 × 4 × 5, 1 × 2 × 3 × 4 × 5, &c. 1 × 2 × 3 × 4 × 5 × 6, &c. firſt of which contains as many Terms as there are Units in Index of b; the ſecond as many as there are Units in the In of c; the third as many as there are Units in the Index of c.

he Demonſtration of this ſee in the above cited Tranſ on.

ere follows an Example or two of the Uſe of this Theo

EXAMPLE I.

05. To raiſe this infinite Series $\frac{1}{x}+\frac{1}{xx}+\frac{1}{x^3}+\frac{1}{x^4}$, &c. e ſecond Power, or to ſquare it.

this Caſe in the Theorem $m=2$, $z=x$, $a=\frac{1}{xx}$, $b=$ $=\frac{1}{x^6}$, $d=\frac{1}{x^8}$, &c. therefore $\overline{\frac{1}{x}+\frac{1}{xx}+\frac{1}{x^3}+\frac{1}{x^4},}^{\,2}$ will be $\frac{1}{xx}+\frac{2}{x^3}+\frac{3}{x^4}+\frac{4}{x^5}$, &c. for the firſt Term $\left(=\frac{1}{x^4}\times xx\right)$ is $=\frac{1}{xx}$. The ſecond Term $\frac{m}{1}a^{m-1}$ $^{+1}\left(=\frac{2}{xx}\times\frac{1}{x^4}\times x^3\right)$ is $=\frac{2}{x^3}$.

$$\text{Third}\ \left.\begin{matrix}\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^2\\ +\frac{m}{1}a^{m-1}c\end{matrix}\right\}z^{m+2}\left\{\begin{matrix}=2\times\frac{1}{2}\times\frac{1}{x^8}\times x^4\\ =\frac{1}{x^4},\end{matrix}\right.$$

$$\text{and }2\times\frac{1}{xx}\times\frac{1}{x^6}\times x^4\text{ is }\left.\begin{matrix}=\frac{1}{x^4},\\ =\frac{2}{x^4}\end{matrix}\right\}=\frac{3}{x^4}.$$

$$\text{Fourth Term}\ \left.\begin{matrix}\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3\\ \frac{m}{1}+\frac{m-1}{1}a^{m-2}bc\\ \frac{m}{1}a^{m-1}d\end{matrix}\right\}z^{m+3}=\left\{\begin{matrix}\\ \\ \\ \end{matrix}\right.$$

2 ×

$$\left.\begin{matrix} 2 \times \overline{\frac{1}{2} \times \frac{0}{3}} \times xx \times \frac{1}{x^{12}} \times x^5 = 0 \\ 2 \times 1 \times 1 \times \frac{1}{x^4} \times \frac{1}{x^6} \times x^5 = \frac{2}{x^5} \\ 2 \times \frac{1}{xx} \times \frac{1}{x^3} \times x^5 = \frac{2}{x^5} \end{matrix}\right\} \text{is} =$$

EXAMPLE II.

606. To ſquare this infinite Series $1 - x + x^2 + x^2 +$ &c.

In this Caſe in the Theorem $m = 2$, $z = x$. $a = \frac{1}{x} -$ $= 1$, $c = -1$, $d = 1$, &c. and ſo $\overline{1 - x + x^2 - x^3 +}$ &c. will be $= 1 - 2x + 3x^2 - 4x^3 + 5x^4$, &c. for a $\left(= \overline{\frac{1}{x} - 1}^{2} \times x^2\right) = 1 - 2x + xx$. The Second T $\frac{m}{1} a^{m-1} b\, z^{m+1} \left(= \frac{2}{1} \times \overline{\frac{1}{x} - 1}^{1} \times x^2\right) = 2xx - 2$

The Third Term $\left.\begin{matrix} \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ \frac{m}{1} a^{m-1} c \end{matrix}\right\} z^{m+2} = \Big\{$

$$\left.\begin{matrix} \frac{2}{1} \times \frac{1}{2} \times \overline{\frac{1}{x} - 1}^{2-2} \times 1 \times x^4 \\ \frac{2}{1} \times \overline{\frac{1}{x} - 1}^{1} \times -1 \times x^4 \end{matrix}\right\} = -x^3 + 2x^4, \text{ \&c.}$$

EXAMPLE III.

607. To raiſe $1 - x + x^3 - x^5 + x^7$, &c. to the Power, or to cube it.

Here $m = 3$. $z = x$. $a = \frac{1}{x} - 1$, $b = 0$, $c = 1$, d and ſo the third Power will be $1 - 3x + 3x^2 + 2x^3 -$ &c. for $a^m z^m \left(= \overline{\frac{1}{x} - 1}^{3} \times x^3\right) = 1 - 3x + 3xx -$

$$\frac{m}{1}a^{m-1}bz^{m+1}\left(= 3 \times \overline{\frac{1}{x}-1}^{\,3-1} \times 0 \times x^4\right) = 0.$$

$$\left.\begin{array}{l}\frac{m}{1} \times \frac{m-1}{2}a^{m-2}b^2 \\ \qquad\frac{m}{1}a^{m-1}c\end{array}\right\}z^{m+2} = \left\{\begin{array}{l} 3 \times 1 \times \overline{\frac{1}{x}-1}^{\,1} \times 0^2 \times x^5 = 0 \\ 3 \times \overline{\frac{1}{x}-1}^{\,2} \times 1 \times x^5 = 3x^3 - 6x^4 \times 3x^5 \end{array}\right\} =$$

$3x^3 - 6x^4 + 3x^5$.

And becauſe $b = 0$, and alſo d, therefore the next Term of the general Theorem will be 0. And thus you may proceed on.

EXAMPLE IV.

608. To extract the Root of an infinite Series; that is, if z be $= ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. to find the Value of x in an infinite Series of Terms affected with z, and free from x.

Firſt, Let us ſuppoſe $x = fz + hz^2 + kz^3 + lz^4 + mz^5 + nz^6$, &c. Then by the Theorem

$$\begin{array}{rl} x^2 = & f^2z^2 + 2fhz^3 \\ & + h^2z^4 + 2hkz^5 + k^2z^6 \\ & + 2fkz^4 + 2flz^5 + 2hlz^6 \\ & \qquad\qquad\qquad + 2fmz^6 \\ & \qquad\qquad\qquad \text{\&c.} \end{array}$$

$$\begin{array}{rl} x^3 = & f^3z^3 + 3f^2hz^4 + 3fh^2z^5 + h^3z^6, \text{ \&c.} \\ & \qquad\qquad + 3f^2kz^5 + 3f^2lz^6 \\ & \qquad\qquad\qquad + 6fhkz^6 \end{array}$$

$x^4 = f^4z^4 + 4f^3hz^5 + 6f^2h^2z^6$, &c.

$x^5 = f^5z^5 + 5f^4hz^6$

$x^6 = f^6z^6$, &c.

Now ſubſtitute theſe Values in the Equation $0 = -z + ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. and then will $-z = -z$.

$$\begin{array}{l} +ax = +afz + ahz^2 + akz^3 + alz^4 + amz^5 + anz^6, \text{ \&c.} \\ +bx^2 = +bf^2z^2 + 2bfhz^3 + bh^2z^4 + 2bflz^5 + bk^2z^6, \\ \qquad\qquad\qquad\qquad + 2bhkz^4 + \\ \qquad\qquad\qquad\qquad\qquad + 2bhkz^5 + 2bfmz^6 \\ \qquad\qquad\qquad\qquad\qquad\qquad + cx^3 \end{array}$$

$$\begin{array}{lllll}
+cx^3 = & * & * & & +cf^3z^3 + 3cf^2hz^4 + cfh^2z^5 + 3cf^2lz^6 \\
 & & & & + 3cf^2kz^5 + \quad \&c. \\
 & & & & 3cf^2lz^6 \\
 & & & & + 6fhkz^6 \\
dx^4 = & * & * & * & + dfz^4 + 4df^3z^5 + 6df^2h^2z^6 \\
 & & & & + 4df^3kz^6 \\
ex^5 = & * & * & * & * \quad + ef^5z^5 + 5ef^4hz^6
\end{array}$$

609. Now if the Sum of the Coefficients of every Term in this Equation be made equal to nothing, we may get the Values of the Coefficients $f, h, k, l, m, n,$ thus. The Sum of the Coefficients of the first Term $\left.\begin{matrix}-1\\af\end{matrix}\right\} z$ will be $af - 1$. Whence if $af - 1 = 0$, then will f be $= \frac{1}{a}$. In like Manner the Sum of the Coefficients of the second Term $\left.\begin{matrix}ah\\bf^2\end{matrix}\right\} z^2$ will be $ah + bf^2$. Whence if $ah + bf^2 = 0$, there will arise $h = \frac{-bf^2}{a} = \frac{-b}{a^3}$. In like Manner the Sum of the Coefficients of the third Term made equal to 0 will be $ak + 2bfh + cf^3 = 0$. Whence $k = \frac{-2bfh - cf^3}{a} = \frac{2b^2 - ac}{a^5}$. Again, $al + bh + 2bfk + 3cf^2h + df^4 = 0$. Whence $l = \frac{-bh - 2bfk + 3cfh - df^4}{a} = \frac{-b^3}{a^7 - 4b^5} + \frac{a^7 + 2ba}{a^6 + 3bc} + \frac{a^6 - d}{a^5} = \frac{5abc - 5b^3 - a^2d}{a^7}$. So likewise m is $= \frac{14b^4 + 6abd - 21ab^2c + 3a^2c^2 - a^3e}{a^9}$. And $n = \frac{-42b^5 + 84ab^3c - 28a^7bc - 28a^2bd + 7a^3cd + 7a^3cd + 7a^3be\,a^4f}{a^{11}}$.

Whence at length substituting these Values of the Coefficients $f, h, k, l, m, n,$ in the assumed Equation $x = fz + hz^2 + kz^3 + lz^4 + mz^5 + nz^6$, &c. and the Root sought will be

$x =$

$$x = \frac{z}{a} - \frac{bz^2}{a^3} + \frac{2b^2 - ac}{a^5}z^3 + \frac{5abc - 5b^3 - a^2d}{a^7}z^4 + \frac{14b^4 + 6a^2bd - 21ab^2c + 3a^2c^2 - a^3e}{a^9}z^5, \text{\&c.}$$

610. If there are any Terms wanting in the propoſed Equation, it is plain that they will likewiſe be wanting in the Root. For Example: If z be $= ax + cx^3 + ex^5$, &c. then will $x = \frac{z}{a} - \frac{ac}{a^5}z^3 + \frac{3a^2c^2 - a^3e}{a^9}z^5$, &c. But let this ſuffice for the preſent; the Application of Algebra to the Solution of Problems in Geometry muſt be deferred till we have premiſed the Principles of that Science, which next follow in a Method not only new, but 'tis preſumed much more conciſe and perſpicuous than any Thing before publiſhed on this Subject.

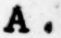
INSTI-

INSTITUTIONS OF GEOMETRY.

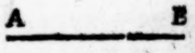
CHAP. I.

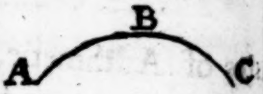
DEFINITIONS.

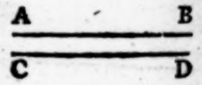
611. A **Point** is that which hath no Parts; and may be repreſented to the Senſes by the Dot A.



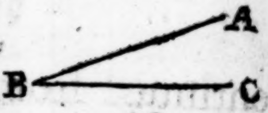
612. A **Line** is deſcribed by the Motion of a Point; and is either a *Right Line* when the Motion is *ſtrait forwards* as A B: Or a *curved Line*, when the Motion is conſtantly directed to different Parts, as the Curve ABC. Lines are ſaid to be *Parallel*, when they are every where at an equal Diſtance, as AB, CD.

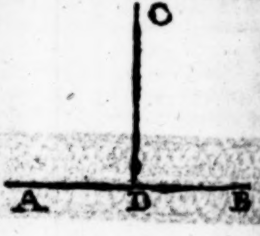




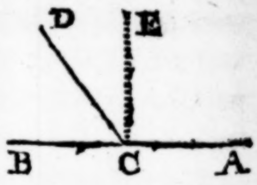


613. An **Angle** ABC is the Space contained between two Lines AB, CB meeting in the Point B. When a Right Line CD, ſtanding upon a Right Line AB, makes the Angles on either Side thereof equal to each other, *viz.* ADC = BDC, then both thoſe equal Angles are *Right Angles*; and the Right Line CD is called a *Perpendicular* to the Line AB.

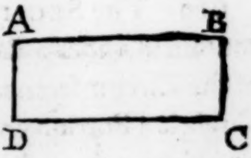




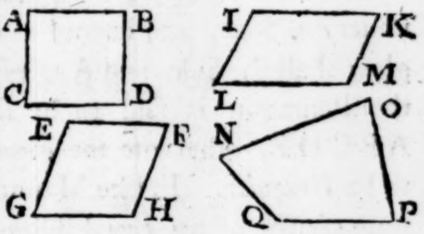
614. An **Obtuse Angle** is greater than a *Right Angle*, as ACD is greater than ACE. And the Angle BCD, which is leſs than the *Right Angle* BCE is called an **Acute Angle**.



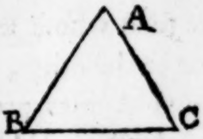
615. A SUPERFICIES is that which has
ngth and *Breadth*, as ABCD, and is
nerated by the Motion of a Line, as of
D or AB carried parallel to itſelf. All
h whoſe 4 oppoſite Sides are equal, and
allel, are called *Parallelograms*. If all the Angles are Right,
h a Figure is called a *Rectangle*.

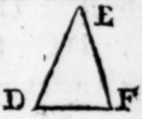


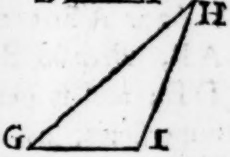
616. If all the 4 Sides are
al, and the Angle right, it
alled a SQUARE, as ABCD.
t if the Angles are not right,
is called a RHOMBUS, as
FGH. A Parallelogram
oſe Angles are not right, is called RHOMBOIDES, as IKLM,
l other *four-ſided Figures* are called TRAPEZIUMS, as the Fi-
re NOPQ.



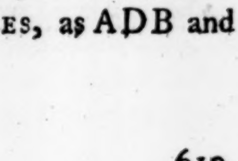
617. A Figure bounded by three Sides is
led a TRIANGLE; if the three Sides be e-
l, it is ſaid to be *Equilateral*, as ABC; if
y two are equal, it has the Name *Iſoſceles*,
DEF; if the three Sides are all unequal,
is ſaid to be *Scalenous*, as GHI. A Trian-
is ſaid to be *Right-angled*, that has one
ght Angle; *Obtuſe-angled*, if it has one An-
Obtuſe; *Acute-angled*, when all the An-
s are Acute; and *Equiangular*, when all
Angles are *equal*.

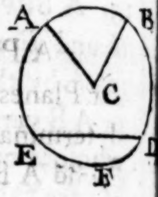




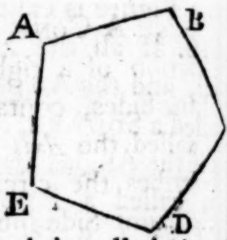


618. A CIRCLE is a Figure contained under
uniform Curve-line ADBE, which is
led the CIRCUMFERENCE or PERIPHE-
, deſcribed about the Point C, which is
led the CENTER, and from which all Lines
wn to the Circumference are equal. The Right Line AB
wn through the Center, is called a *Diameter*; and divides the
rcle into two equal Parts called SEMICIRCLES, as ADB and
EB.

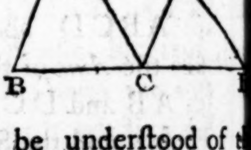




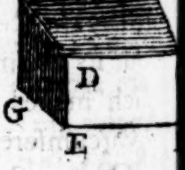
619. The SEGMENT of a Circle is a Figure contained under a Right-line E D, and a Portion of the Circumference E F D. A SECTOR of a Circle is a Portion contained between two Right-lines drawn from the Circumference to the Center, as A C B.



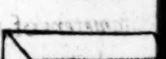
620. A *Pentagon* is a Figure contained under *five Sides*, and having as many Angles; if all the Sides and Angles are equal, the Pentagon is ſaid to be *Regular*, as A B C D E, otherwiſe the Figure is ſaid to be *Irregular*. In like Manner, if a Figure conſiſt of Six equal Sides and Angles, it is called a *regul* HEXAGON; if of *Seven*, it is an HEPTAGON; if of Eight OCTAGON, and ſo on.



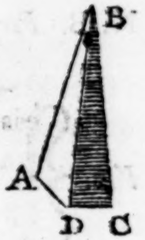
621. SIMILAR Right-lined Figures are ſuch whoſe ſeveral Angles are equal one to the other, and the Sides about the equal Angles proportional. Thus the Angle B = D C E, and the Sides A B : B C : : D C : C E. And the ſame is to be underſtood of t other Angles and Sides reſpectively.



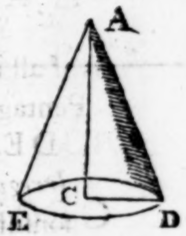
622. A SOLID is that which hath *Length*, A B, *Breadth* B C, and *Thickneſs* or *Depth* D E; and is generated by the Motion of a Superficies, as A B C D carried parallel to E F, or of A D E G to C F. When all theſe Dimenſions are unequal the Solid is called a *Parallelopipedon*; but when they are all equal, it is called a CUBE, as A B, which is bounded by Six equal Squares.



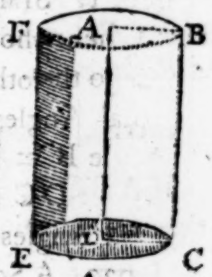
623. A PRISM is a Solid contained under Planes, whereof the two End ones are equal, alike, and parallel; but the Sides are Parallelograms.



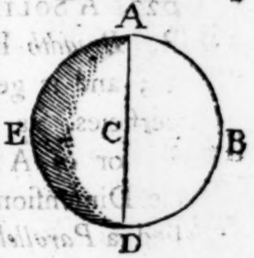
624. A **Pyramid** is contained under divers triangular Planes ſet upon one Plane (called the *Baſe*) and terminating at the other End in one Point; as the ſolid ABCD.



625. A **Cone** is a Solid generated by the Revolution of a right-angled Triangle about one of the Sides, containing the Right-angle, which is called the *Axis* of the Cone; the other Side deſcribes the circular *Baſe* of the Cone; while the ſlaunt Side ſubtending the Right-angle, deſcribes the *Conic Superficies.* Thus ACD is the revolving Right-angled Triangle; and ADE the generated Cone.



626. A **Cylinder** is a Solid generated by the Revolution of a right-angled Parallelogram ABCD about its Side AD, which makes the *Axis* of the Cylinder; while the too Sides AB and DC deſcribe the circular Baſes or Ends; and the Side BC the curved Surface thereof.



627. A **Sphere** is a ſolid ABDE generated by the Revolution of Semicircle ABD about its Diameter AD (remaining fixed) which makes the *Axis* of the Sphere while the Circumference of the Semicircle deſcribes the Convex Superficies of the Sphere.

628. *Similar Solids* are thoſe which are contained under *ſimilar Planes* equal in Number. Hence all *Cubes* and *Spheres* are *ſimilar Solids.* And all *Cones,* and *Cylinders* are *ſimilar,* whoſe *Axis* and *Diameters* of their Baſes are *proportional.*

N. B. *The following Theorems do much depend upon the Axioms in* Inſt. 198, 199, 200, 201, 202; *and therefore they ought to be well remembered by the Reader.*

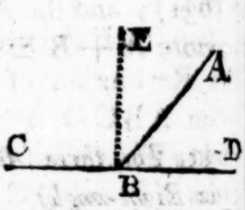
N. B. *For the Sake of Conciſeneſs and Perſpicuity, the following Symbols or Characters are uſed.*

Viz.
- ∟ Denotes a *right Angle.*
- ‖ Parallel to.
- □ A Square.
- ▭ Parallelogram.
- △ A Triangle.

CHAP. II.

GEOMETRICAL THEOREMS.

THEOREM I.

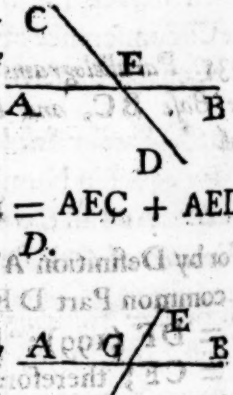


629. *A Right-line* AB *ſtanding upon another* CD, *makes Angles* ABC + ABD = 2 *Right-angles.*

DEMONSTRATION.

If ABC = ABD, then is each a Right-angle by (613). If they are unequal; let BE be perpendicular to CD; then is ABD = ∟ — ABE, and ABC = ∟ + ABE, therefore (175) ABD + ABC = 2 ∟. *Q. E. D.*

THEOREM II.

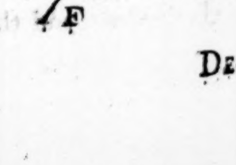


630. *Two Right-lines* AB, CD, *interſecting each other in* E, *make the two oppoſite Angles equal, viz.* AED = CEB.

DEMONSTRATION.

The Angle AEC + CEB = (629) 2 ∟ = AEC + AED; therefore BEC = AED (200). *Q. E. D.*

THEOREM III.



631. *A Right-line* EF *cutting two Parallel Lines* AB, CD, *makes the alternate Angles equal, viz.* AGH = DGH.

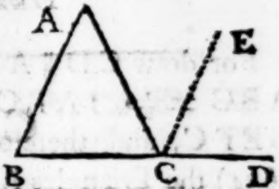
DE-

DEMONSTATION.

Thus AGE + AGH = (629) 2 ∟ = CHG + GHD; but becauſe of parallel Lines, it is AGE = CHG; therefore AGH = GHD. *Q. E. D.*

THEOREM IV.

632. *The outward Angle* ACD *of any Triangle* ABC, *is equal to the two internal oppoſite Angles* A *and* B.



DEMONSTRATION.

Let CE be parallel to AB; then is the Angle A = ACE by (631); and the Angle B = ECD, becauſe of AB ∥ EC; therefore A + B = ACE + ECD = ACD. *Q. E. D.*

THEOREM V.

633. *The three Angles of every plain Triangle* ABC, *are equal to two Right-angles.*

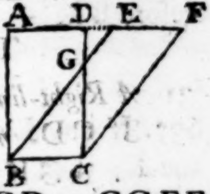
DEMONSTRATION.

For ACD + ACB = 2 ∟, by (629); and ACD = A + B, by (632); therefore ACB + A + B = 2 ∟. *Q. E. D.*

THEOREM VI.

634. *Parallelograms* ABCD, BEFC, *which ſtand upon the ſame Baſe* BC, *and between the ſame Parallels* AF, BC, *are equal.*

DEMONSTRATION.

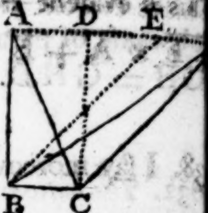


For by Definition AD = BC = EF; add the common Part DE to both, and we have AE = DF (199); but AB = DC, and BE = CF; therefore is the Triangle ABE = DCF; take away the common Triangle DGE, there will remain the Trapezium ABGD = CGEF; to each of theſe add the Triangle BCG, and it makes the Parallelogram ABCD = BEFC. *Q. E. D.*

THE-

THEOREM VII.

635. *Triangles* ABC, BFC, *ſtanding upon the ſame Baſe* BC, *and between the ſame Parallels* AF *and* BC, *are equal to one another.*



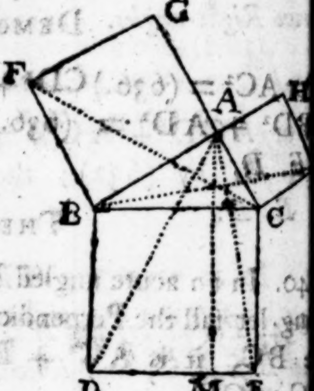
DEMONSTRATION.

For draw $CD \parallel AB$, and $BE \parallel CF$; then is the Triang $ABC = \frac{1}{2}\square ABCD$, and the Triangle $BCF = \frac{1}{2}\square$ BEFC, and therefore ſince the Parallelograms are equal b (634) the Triangles ABC, BCF will be ſo too. *Q. E. D.*

THEOREM VIII.

636. *In a Right-angled Triangle* ABC, *the Square* BE *ma upon the Side* BC *ſubtending the Right-angle* BAC, *is equal to t Sum of the Squares* BG, CH, *which are made of the Sides* AB *and* A

DEMONSTRATION.



Join AE, and AD, and draw $AM \parallel CE$, becauſe the Angle $DBC = FBA$ (616), add to both the common Angle ABC, then is the Angle $ABD = FBC$. Again, $AB = FB$, and $BD = BC$, (616); therefore the Triangle $ADB = FBC$. But the Parallelogram $BM = 2\,ABD$; and the Parallelogram $BG = 2\,FBC$, as is evident from the Figure to (635). Therefore is the $\square\, BM = BG$ the Square of the Side A In the ſame Manner, it is ſhewn that the $\square\, MC = CH$ therefore $BM + CM = BE = BG + CH$. *Q. E. D.*

THEOREM IX.

637. If any Line z be divided into two Parts A and E; then it is $zz = A^2 + E^2 + 2AE$.

DEMONSTRATION.

This is evident from (186). For $z = A + E$; and $zz =$ $\overline{+E} \times \overline{A + E} = A^2 + 2AE + E^2$. *Q. E. D.*

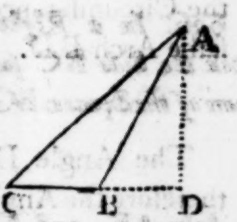
THEOREM X.

638. I ſay moreover, it is $z^2 + E^2 = A^2 + 2zE$.

DEMONSTRATION.

For to the above Equation $z^2 = A^2 + 2AE + E^2$, add both Sides E^2, and it is $z^2 + E^2 = A^2 + 2AE + 2E^2$ $A^2 + 2E \times \overline{A + E} = A^2 + 2EZ$. *Q. E. D.*

THEOREM XI.

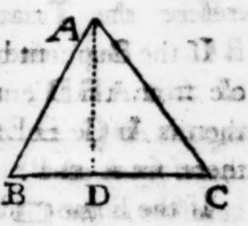


39. In an obtuſe angled Triangle ABC, ng let fall the Perpendicular AD to the C B, continued out, it is $AC^2 = CB^2$ $AB^2 + 2CB \times BD$.

DEMONSTRATION.

For $AC^2 =$ (636.) $CD^2 + AD^2 =$ (637.) $CB^2 + 2\overline{BC \times BD}$ $BD^2 + AD^2 =$ (636.) $CB^2 + 2CB \times BD + AB^2$. *E. D.*

THEOREM XII.



40. In an acute angled Triangle ABC, ng let fall the Perpendicular AD on the BC, it is $AC^2 + BC^2 = AB^2 +$ $C \times CD$.

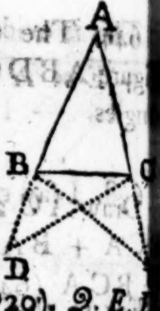
DEMONSTRATION.

or $AC^2 + BC^2 =$ (636.) $AD^2 + DC^2 + BC^2 =$ (638.) $+ BD^2 + \overline{2BC \times CD} =$ (636.) $AB^2 + 2\overline{BC \times CD}$. *E. D.*

THEOREM XIII.

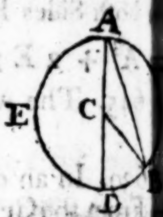
41. In an Iſoſceles Triangle ABC, the equal Sides AB, ſubtend equal Angles ABC, ACB at the Baſe BC.

DEMONSTATION.



Produce A C, A B, and make A E = A D; and join B E, C D; then becauſe A C = A B, and A E = A D and the common Angle A, the Triangles A B E, A C D will be equal, and the Angle E = D; and the Baſe B E = C D; alſo E C = B D, by Conſtruction, therein the Triangles B E C, B D C, the Angle E C B = D B C; and conſequently the Angle A C B = A B C by (629). *Q. E.*

THEOREM XIV.



642. In a Circle A B D E, the Angle B C D, at the Center C, is double of the Angle A at the Circumference, when they ſtand upon the ſame Arch B D.

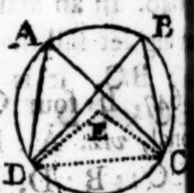
DEMONSTRATION.

The Angle D C B = A + B; but A C = C B (by 618) therefore the Angle A = B (by 641); conſequently, the Ang D C B = 2 A. *Q. E. D.*

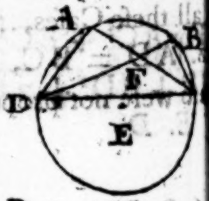
THEOREM XV.

643. Any two Angles D A C, D B C, in the ſame Segme of a Circle D A B C, are equal.

DEMONSTRATION.



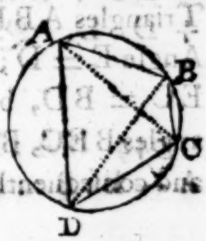
If the Segment be greater than a Semicircle from the Center E, draw E D, E C; then is Angle D E C = (642.) 2 A = 2 B, therefore A = B. *Q. E. D.*



If the Segment be leſs than a Semicircle; then in the Triangles D A F, B D F, the Sum of the Angles in each is the ſame (633). From each of which, take away the Angle A F D = B F C (650.) and A D B = B C A, by the firſt Caſe of this. There will remain the Angle D A C = D B C. *Q. E. D.*

THEOREM XVI.

644. The two oppoſite Angles D and B of any four-ſided igure ABDC deſcribed in a Circle, are equal to two Right ngles.



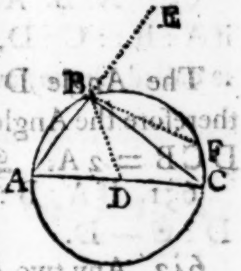
DEMONSTRATION.

Draw AC and BD; the Angle ABC + CA + BAC = 2 ∟ (633). But BDA =BCA (643), and alſo BDC = BAC; herefore ABC + ADC = 2 ∟. *Q. E. D.*

THEOREM XVII.

645. The Angle ABC in a Semicircle is a Right one.

DEMONSTRATION.



From the Center D draw DB; becauſe DB =DA, the Angle A = DBA, and the ngle DCB = DBC (641); therefore the ngle ABC = B + ACB = EBC (632); herefore the Angles ABC and EBC are ight-ones, by (613). *Q. E. D.*

646. *Corol.* Hence the Angle A in the greater egment BAC, is leſs than a Right Angle, ecauſe A + C = ∟ (by 633 and 645); and the Angle ABF the leſſer Segment ABF, is greater than a Right Angle BC.

THEOREM XVIII.

647. If four Quantities A, B, C, D, are *directly* propor- onal, *viz.* A : B :: C : D, they will be ſo *alternately*, viz. :C :: B : D, as alſo *inverſely*, viz. B : A :: D : C. For all theſe Caſes, the Products of *Extremes and Means are equal*, iz. AD = BC, which could not be if the Terms in each ſe were not proportional by (522).

THEOREM XIX.

648. If A : B :: C : D, then by *Compoſition* of Ratios, it is +B : B :: C + D : D.

DEMONSTRATION.

For ſince $A : B :: C : D$, it is $AD = BC$ (322); add
both Sides the Quantity DB, and the Equation is $AD + D$
$= BC + DB$, which gives this Analogy, $A + B : B ::$
$+ D : D$. *Q. E. D.*

THEOREM XX.

649. If $A : B :: C : D$, then by Diviſion of Ratios, it
$A - B : B :: C - D : D$.

DEMONSTRATION.

For ſince $A : B :: C : D$, it is $AD = BC$ (322); fro
each Side ſubduct DB, and the Reſidue is $AD - DB = B$
$- DB$; wherefore $A - B : B :: C - D : D$. *Q. E. D.*

650. *N. B.* After the ſame Manner it may be ſhewn, th
if $A : B :: C : D$, it is alſo $A : B \pm A :: C : C \pm D$, whi
is called *Converſion of Ratios*.

THEOREM XXI.

651. If $A : B :: C : D$, it will be $A + B : A - B :: C$
$D : C - D$.

DEMONSTRATION.

For $AD = BC$ (322) and $2AD = 2BC$; to which
each Side add $AC - DB$, and we have $AC + 2AD$
$DB = AC + 2BC - DB$, that is, $AC + CB - A$
$- DB = AC + AD - CB - DB$, that is, $\overline{A + B}$
$\overline{C - D} = \overline{A - B} \times \overline{C + D}$. Conſequently $A + B : A$
$B :: C + D : C - D$. *Q. E. D.*

THEOREM XXII.

652. If there be two Rows of Magnitudes $\left\{\begin{array}{l} A, B, C, \text{\&c.} \\ R, S, T, \text{\&c.} \end{array}\right.$

which taken ordinately two and two, are in the ſame Ratio, *viz.* $\left\{\begin{array}{l} A : B :: R : S \\ B : C :: S : T, \end{array}\right.$

They ſhall be alſo in the ſame Ratio by Equality, *viz.* $\left\{ A : C :: R : T. \right.$

DEMONSTRATION.

For we have AS = BR, and BT = CS by Hypotheſis. Then AS × BT = BR × CS; divide each Side by BS, there remains AT = RC, which gives A : C :: R : T. *Q. E. D.*

Corol. Hence alſo A : R :: C : T.

THEOREM XXIII.

653. Ratios A : B and C : D, which are the ſame to any third Ratio *x* : *y*, are the ſame to one another.

DEMONSTRATION.

Since A : B :: *x* : *y*, it is A*y* = B*x* (322); alſo becauſe C : D :: *x* : *y*, it is C*y* = D*x*; and therefore A*y* × D*x* = C*y* × B*x*; divide each Part by *yx* and there remains AD = CB; whence A : B :: C : D. *Q. E. D.*

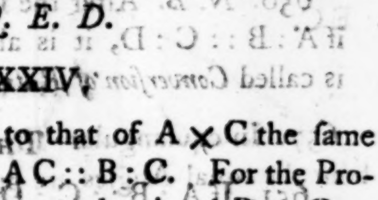
THEOREM XXIV.

654. The Ratio of A × B is to that of A × C the ſame with the Ratio of B to C, or AB : AC :: B : C. For the Product of the Extremes and Means are equal, *viz.* AB × C = AC × B. See (322).

THEOREM XXV.

655. Parallelograms, *acde*, *bcgf*, which have the ſame Height *ic*, have the ſame Ratio to one another as their Baſes, *ed*, *gf*, have.

DEMONSTRATION.



The ▭ *acde* = ▭ *acik*, and *cbgf* = *cbhi*, by (634). Let *bh* = *ci* = *ak* = A; *ib* = *gf* = B; and *ki* = *ed* = C. Then is the ▭ *cbhi* = A × B, and the ▭ *acik* = A × C. Therefore (654) *cbhi* : *acik* :: AB :: AC : B : C :: *ib* : *ik* :: *gf* : *ed* : *cbfg* : *acde*. *Q. E. D.*

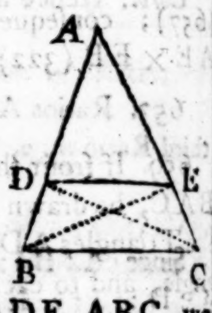
Corol. The ſame thing is true of the Triangles *cfg* and *ced*, as being the Halves of the Parallelograms, and which is evident alſo by (635).

THE-

THEOREM XXVI.

656. If in the Triangle ABC, you draw DE ∥ BC, then will the Sides AB, AC, be cut proportionally, or it will be AD : DB : : AE : EC.

DEMONSTRATION.



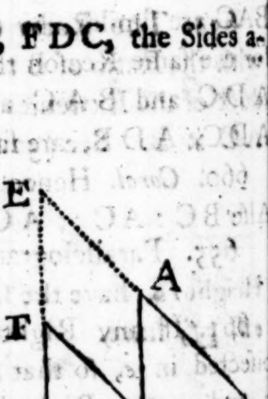
Join BE, and DC; becauſe the Triangle DEB = DEC by (635), we have the Triangle ADE : DBE : : ADE : ECD; but (by 655) the Triangle ADE : DBE : : AD : DB; and ADE : DEC : : AE : EC. Therefore (653) it is AD : DB : : AE : EC. *Q. E. D.*

Coral. Hence becauſe of ſimilar Triangles ADE, ABC, we have AD : AE : : AB : AC; and therefore AB : AC : : DB : EC.

THEOREM XXVII.

657. In equiangular Triangles ABD, FDC, the Sides about the equal Angles are proportional.

DEMONSTRATION.



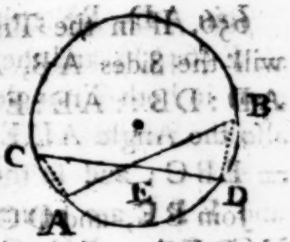
Let their Baſes CD and DB make a right Line CB; and produce AB and FC till they meet in E; becauſe the Angle C = ADB, therefore CE ∥ AD by (631). Alſo becauſe the Angle CDF = ABD, it is AB ∥ FD; therefore AEFD is a Parallelogram, and conſequently EF = AD (615), and FD = AE. Whence FC : FE (= AD) : : CD : DB by (656); therefore by Alternation it is FC : CD : : AD : DB (647). Alſo CD : DB : : AE (= FD) AB; whence again by Alternation, CD : FD : : DB : AB. Wherefore by Equality (652) it is FC : FD : : AD : AB. *Q. E. D.*

THEOREM XXVIII.

658. If in a Circle two right Lines AB, CD, interſect each other in E, the Rectangle AE × EB ſhall be equal to the Rectangle CE × ED.

DE-

DEMONSTRATION.

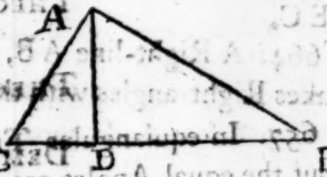


Join AC and DB; then the Angle CEA = BED (630); alſo the Angle C = B (643); therefore the Triangles AEC, and DEB are equiangular (633); whence it is CE : EA :: EB : ED (657); conſequently CE × ED = AE × EB (322). *Q. E. D.*

THEOREM XXIX.

659. If from the Right-angle A, of a right-angled Triangle BAC, be drawn the Perpendicular AD to the Baſe, then will the Triangles ADB and ADC be ſimilar both to the whole Triangle, and to each other.

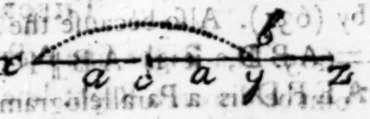
DEMONSTRATION.



Becauſe the Angle BAC = ADB = ∟ (613); and B common to both, the Triangles ADB and BAC are ſimilar, by (621, 657); for the ſame Reaſon the Triangles ADC and BAC are ſimilar. Conſequently the Triangles ADC, ADB, are ſimilar to each other. *Q. E. D.*

660. *Corol.* Hence it is BD : DA :: DA : DC by (621). Alſo BC : AC :: AC : DC; and CB : BA :: BA : BD.

THEOREM XXX.



661. If any Right-line xy be biſected in c, ſo that $xc = cy = a$, and another Right-line $yz = b$ be added thereto; then $\overline{2a + b} \times b + aa = aa + 2ba + bb = \overline{+ b}^2$.

DEMONSTRATION.

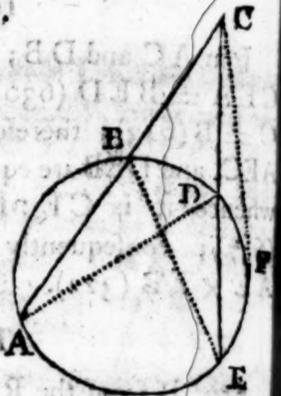
This is evident from (186), or thus $\overline{xz + yz} \times yz + cy^2 = \overline{y + yz}^2$.

THEOREM XXXI.

662. If from any Point C without a Circle, are drawn two Right-lines CA, CE, cutting the Circle in B and D, then it will be AC × BC = EC × DC.

DE-

DEMONSTRATION.



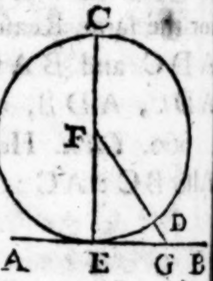
Join AD, BE; then is the Angle A = E (643); and the Angle C is common in both Triangles ADC, EBC; alſo the Angle ADE = ABE, ADC = EBC; and ſo the Triangles ADC and EBC are equiangular; therefore DC : CA :: BC : CE (657); whence AC × BC = EC × DC (61).
Q. E. D.

663. *Corol.* Let EC remove into the Situation CF, touching the Circle in F; then will EC = DC = CF; and the Theorem will become $AC \times BC = CF^2$; whence in this Caſe, AC : CF :: CF : BC.

THEOREM XXXII.

664. A Right-line AB, touching a Circle in the Point E, makes Right-angles with the Diameter CE.

DEMONSTRATION.

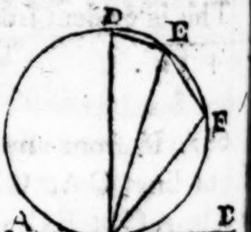


If it be denied, let the Line FG make Right-angles therewith. Then the Side FE which ſubtends the Right-angle FGE, and which is equal to FD, is greater than the Side FG (636), *which is abſurd.*

THEOREM XXXIII.

665. If a Right-line AB touch a Circle in C, and from that Point be drawn a Right-line CE cutting the Circle, the Angles ECB, ECA, which it makes with Tangent-line, are equal to the Angles EDC, EFC, which are made in the alternate Segments of the Circle.

DEMONSTRATION.



Let CD be a Diameter or Perpendicular to AB, then the Angle CED is right (645); therefore the Angle D + DCE = ∟ = ECB + DCE; therefore the Angle D = ECB; the firſt Thing.
Q. E. D.

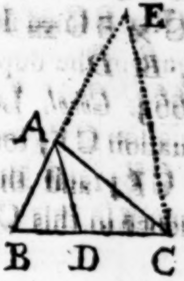
Again, becauſe the Angle ECB + ECA = 2 L (626) = + F (644), from both take away ECB = D (above), and ere will remain ECA = F; the ſecond Thing. *Q. E. D.*

THEOREM XXXIV.

666. If any Angle A of a Triangle BAC be biſected, the ight-line AD that biſects it ſhall divide the Baſe BC, ſo that D : DC :: AB : AC.

DEMONSTRATION.

Produce AB, and make AE = AC, and join E; then becauſe AE = AC, the Angle ACE E (641) and ACE + E = BAC (632) d becauſe BAD = DAC, therefore BAD E; hence DA ∥ CE, (631) wherefore BA AE (= AC) :: BD : BC (656). *Q. E. D.*

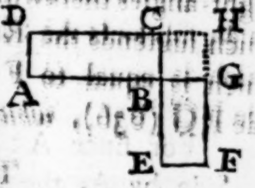


THEOREM XXXV.

667. Equal Parallelograms, having one Angle in each equal, z. ABC = EBG, have the Sides about the equal Angles ciprocally proportional.

DEMONSTRATION.

Let the Sides AB, BG, about the equal ngles make one Right-line, then ſhall B, BC do the ſame, and produce FG, C, till they meet. Then it is AB : BG BD : BH (655) :: BF : BH :: EB : C. *Q. E. D.*



668. *Corol.* Hence the ſame Thing is evident in *Triangles* alſo, being the Halves of the Parallelograms. From this Theorem pears the Reaſon of *reſolving Equations into Analogies.* For let B = a, BG = b, EB = c, BC = d. Then by Hypo-eſis $ad = cb$; and by the Theorem, $a : b :: c : d$.

THEOREM XXXVI.

669. Similar Parallelograms A × B and C × D are in a Du-cate Ratio, or as the Squares of their like Sides.

De-

DEMONSTRATION.

Since $A : B :: C : D$, therefore $AD = BC$; multiply this Equation by the Rectangle BD, it produces $AB \times D^2 = CD \times B^2$; whence $AB : CD :: B^2 : D^2$, (668). *Q. E. D.*

670. *Corol.* Hence all ſimilar Triangles, and all regu Right-lined ſimilar Figures which are reſolvable into Triangl are in the duplicate Ratio of their like or homologous Sides.

THEOREM XXXVII.

671. If four Quantities are proportional, their Squares ſh be ſo.

DEMONSTRATION.

Let the Quantities be A, B, C, D; and ſince $A : B ::$ D, it is $AD = BC$; multiply this Equation by itſelf, it p duces $A^2 B^2 = B^2 C^2$; wherefore $A^2 : B^2 :: C^2 : D^2$ (32 *Q. E. D.*

THEOREM XXXVIII.

672. If three Quantities A, B, C, are proportional, it ſh be $A : C :: A^2 : B^2$.

DEMONSTRATION.

For ſince $A : B :: B : C$, it is $AC = B^2$; multiply e Side by A, the Product is $A^2 C = B^2 A$; therefore $A : C$ $A^2 : B^2$. *Q. E. D.*

673. *Scholium.* If the ſame Equation $AC = B^2$ be multi ed by B, it produces $ACB = BBB$, whence $B : C :: A$ B^2. Conſequently $A : B :: AB : BB$. That is, *any two N bers have the ſame Ratio, as the Rectangle made of them has to Square of the Conſequent.*

THEOREM XXXIX.

674. If four Quantities A, B, C, D, are proportiona will be $A : D :: A^2 : BC$.

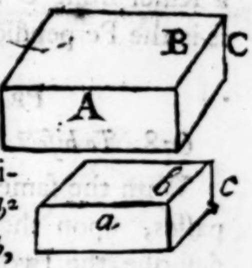
DEMONSTRATION.

Since $A : B :: C : D$, it is $AD = BC$; which multiplied on both Sides by A, gives $A^2D = ABC$; therefore $A : D :: AA : BC$. *Q. E. D.*

THEOREM XL.

675. Similar Solids are in the *Triplicate Ratio*, or as the Cubes of their like Sides.

DEMONSTRATION.



Let the three Dimenſions of one Solid be A, B, C, and of the other a, b, c, reſpectively. Then ſince $A : a :: B : b$, it is $Ab = aB$. Again, $B : b :: C : c$, therefore $Bc = Cb$; let theſe Equations be multiplied together, they make the Product $ACb^2 = B^2ac$; multiply this on each Side by Bb, and it produces $ABCb^3 = B^3abc$; whence $ABC : abc :: B^3 : b^3$. *Q. E. D.*

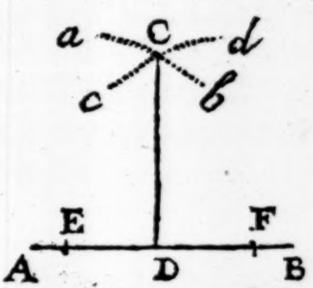
Theſe forty Theorems will ſuffice for our Purpoſe at preſent, to which we ſhall add the following Problems neceſſary in *Practical Geometry*.

CHAP. III.

GEOMETRICAL PROBLEMS.

PROBLEM I.

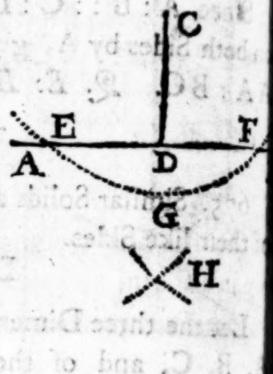
676. UPON *a given Right-line* AB *to erect a Perpendicular* CD, *in the Point* D.



Take on either Side the given Point D, the Diſtance $DE = DF$; with any opening of the Compaſſes, ſet one Foot in E, and F, and ſtrike the Arches ab, and cd, cutting each other in C; join CD and it is the Perpendicular required.

PROBLEM II.

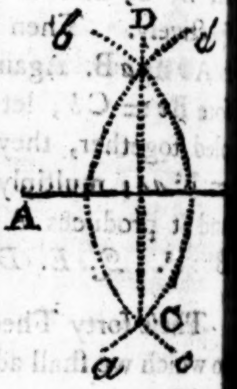
677. *Upon a Right-line* AB *from any given Point* C *above it, let fall the Perpendicular-line* CD.



On the given Point C, ſtrike the Arch of a Circle EGF interſecting the Line AB in E and F; on the Points E and F, make the Interſection at H; lay a Ruler from C to H, and draw CD, it is the Perpendicular required.

PROBLEM III.

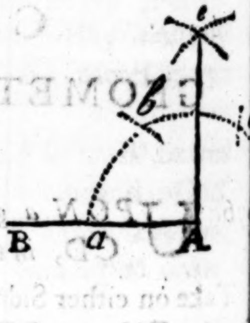
678. *To biſect a given Right-line* AB.



With the ſame Opening of the Compaſſes, upon the Extremities A and B, deſcribe the two Arches *ab*, *cd*, cutting each other in C and D; thro' the Points C and D, draw the Right-line CD, and it will biſect the given Line AB, as required.

PROBLEM IV.

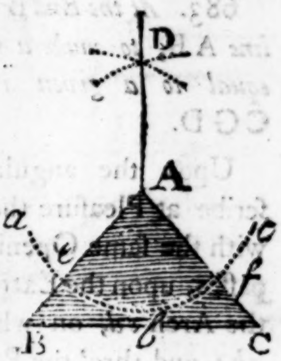
679. *To erect a Perpendicular on the Extremity* A *of a gi*
Right-line AB.



On the Point A deſcribe the Arch *ad*; and with the ſame Opening of the Compaſſes, from *a* make the Interſection *b*, and on *b*, the Interſection *c*; then from *b* and *c* make the Interſection *e*; and draw *e*A the Perpendicular required.

PROBLEM V.

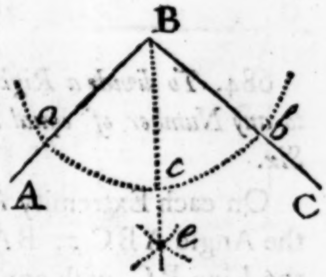
680. *On the Point* A *of a given Angle* BAC *to erect a Right-line* AD, *which ſhall incline neither to the Right-hand nor to the Left.*



On the angular Point A deſcribe the Arch *abc*, cutting the Sides in *e* and *f*; on *e* and *f* make the Inſection D, thro' which draw the Line AD, and it is done.

PROBLEM VI.

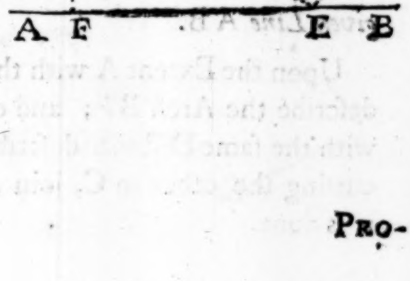
681. *To biſect a Right-lined Angle* ABC.



On the Point B deſcribe the Arch *acb* cutting the Sides in *a* and *b*, on which Points makes the Interſection *e*, and draw the Line *e*B; and the Angle ABC is biſected thereby.

PROBLEM VII.

682. *Through a given Point* D, *to draw a Right-line* DC *parallel to a given Right-line* AB.

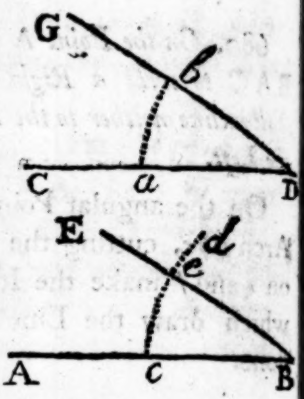


From the Point D draw at Pleaſure the Oblique-line DE; on the Points D and E, deſcribe the Arches E*b*, and F*a*; and make the Arch $EC = FD$; and then through the Points D, and C, draw the Line DC; and it will be parallel to AB.

PRO-

PROBLEM VIII.

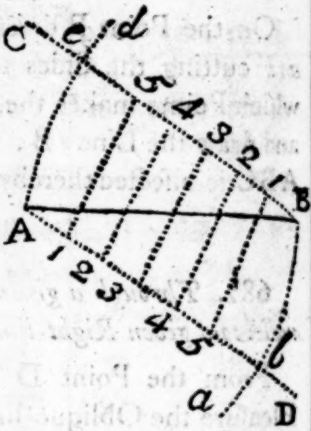
683. *At the End* B *of a given Right-line* A B, *to make a right-lined Angle equal to a given right-lined Angle* CGD.



Upon the angular Point D deſcribe at Pleaſure the Arch *a b*; and with the ſame Opening of the Compaſſes, upon the Extremity B deſcribe the Arch *c d*, on which make *c e* = *a b*; and thro' the Points B, *e*, draw the Line E B, and it will make the Angle ABE = CDG.

PROBLEM IX.

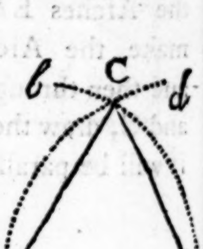
684. *To divide a Right-line* A B *into any Number of equal Parts, ſuppoſe Six.*



On each Extremity A and B make the Angle ABC = BAD; then on the Line B C with any ſmall Opening of the Compaſſes make the Five equal Diſtances, at 1, 2, 3, 4, 5, and alſo do the ſame on the Line A D, then draw the Lines 5 1, 2 4, 3 3, &c. as in the Figure, and they ſhall divide the Line A B in ſix equal Parts as required.

PROBLEM X.

685. *To make an equilateral Triangle upon a given Line* A B.

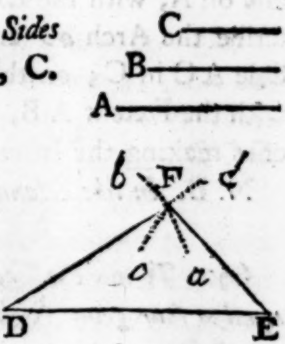


Upon the Extent A with the Diſtance A B, deſcribe the Arch B *b*; and on the Extent B with the ſame Diſtance deſcribe the Arch A *d*, cutting the other in C, join A C, B C, and it is done.

PRO-

PROBLEM XI.

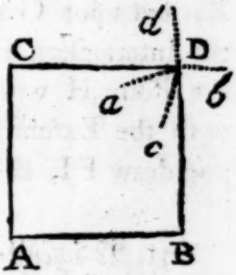
686. *To make a Triangle whoſe three Sides ſhall be equal to three given Right-lines* A, B, C.



Make DE = A, and on D with the Extent of the Line B deſcribe the Arch *ab*; and on E, with the Extent of the Line C, deſcribe the Arch *cd*, cutting the former in F; join DF, EF, and it is done.

PROBLEM XII.

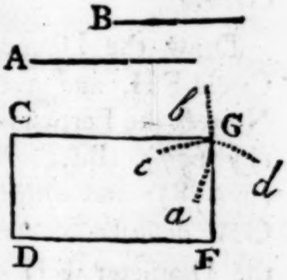
687. *To make a Square upon a given Right-line* AB.



On the Extent A erect the Perpendicular AC = AB (679); with the Extent AB on the Points B, C, deſcribe the Arches *ab*, *cd*, interſecting in D; draw CD, BD, and it is done.

PROBLEM XIII.

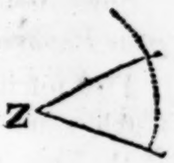
688. *To make a Parallelogram, whoſe two unequal Sides are equal to two given Right-lines* A, B.

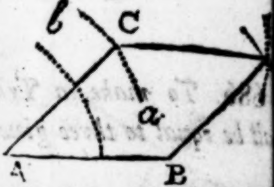


Make DE = A; and on the Extent D, erect the Perpendicular DC = B; then on the Point C, with the Extent of A, deſcribe the Arch *ab*; and on F, with the Extent of B, deſcribe the Arch *cd*, interſecting the former in G; draw CG, FG, and it is done.

PROBLEM XIV.

689. *To make a Rhombus* on the Right-line AB whoſe acute Angle ſhall be equal to a given right-lined Angle Z.



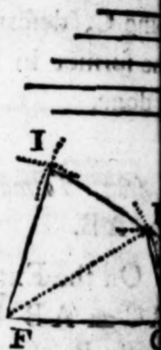


Make the Angle A = Z (by 683) and on A, with the Extent of A B, deſcribe the Arch *a b* and it will cut the Side A C in C; on the Points C and B with the Extent A B, deſcribe the Arches making the Interſection D; join C D, B D, and it is don

N. B. *In like Manner you make a given Rhomboides.*

PROBLEM XV.

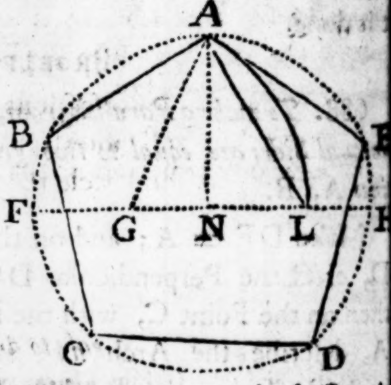
690. *To make a Trapezium whoſe Sides ſhall be equal to four given Right-lines* A, B, C, D, *and whoſe Diagonal is equal to a Right-line* E.



Make F G = A; upon F with the Extent of E, and upon G with the Extent of D, make the Interſection H, and draw G H. Then on the Point H with the Extent of C, and on F with the Extent of B, make the Interſection I; and draw F I, H I, and it is done.

PROBLEM XVI.

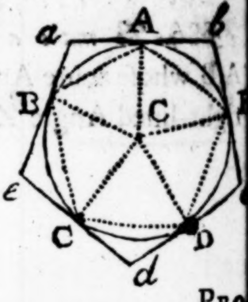
691. *To inſcribe an equilateral and equiangular Pentagon in a Circle.*



Draw the Diameter of the Circle F H, and on the Center N erect the Perpendicular A N (by 677). Biſect N F in G (by 678); and with the Radius G A, deſcribe an Arch cutting the Diameter F H in L, and draw A L; it ſhall be the Side of the Pentagon required;* vi A L = A B = B C = C D = D E = E A.

PROBLEM XVII.

692. *About a given Circle to deſcribe a regular Pentagon.*



Firſt inſcribe a regular Pentagon in the Circle; and from the Center C draw the Right-lines C A, C B, C D, &c. to the ſeveral Angles thereof; and to theſe Lines draw Perpendiculars *a b*, *a c*, *e d*, *d c*, *c b*, and it is done.

PRO

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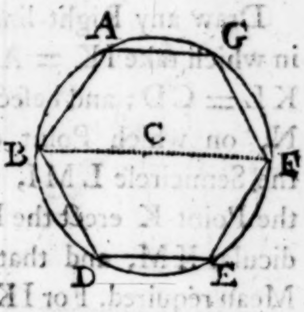
* See Dr. *Barrow*'s Scholium to Prop. X. Book XIII. of *Euclid.*

PROBLEM XVIII.

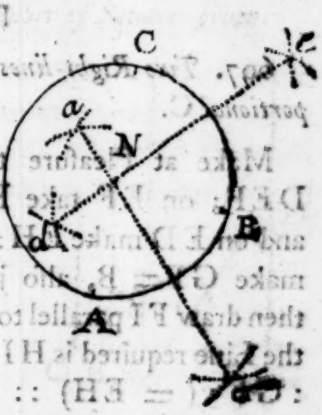
693. *To inſcribe a regular Hexagon in a Circle.*



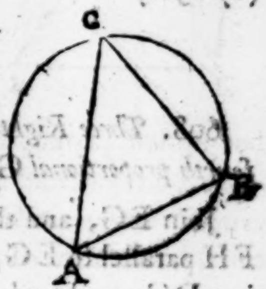
Draw the Diameter B F, the Half of
hich, B C, ſet round upon the Cir-
le gives the ſeveral Points, A, B, C,
D, E, F, G; which joined, conſti-
ute the Hexagon required.

PROBLEM XIX.

694. *To find the Center of a Circle.*



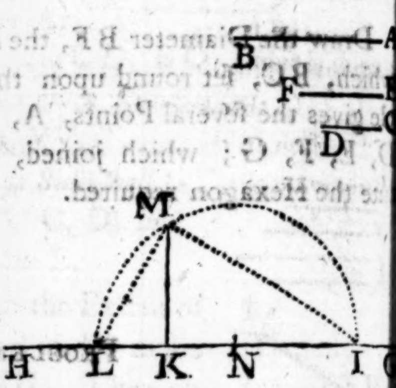
Take any three Points A, B, C,
the Circumference, and upon A and
, with the ſame Opening of the Com-
ſſes make the Interſections *a* and *b*,
d draw the Line *a b*. On the Points
and C, make the Interſections *d*, *e*,
d draw *d e*, it will interſect *a b* in
, the Center of the Circle that was
be found.



695. N. B. *Hence it is eaſy to draw a Cir-*
through any three Points given, A, B, C.
any Part of a circular Arch A B C *may*
compleated into a Circle. And thus alſo
a Circle be circumſcribed about the given
iangle A, B, C.

PROBLEM XX.

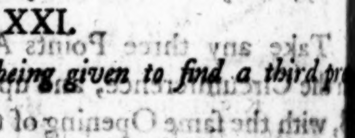
696. *To find a mean proportional* EF *between two given Right-lines* AB, CD.



Draw any Right-line GH, in which take IK = AB, and KL = CD; and beſect IL in N, on which Point deſcribe the Semicircle LMI, then on the Point K erect the Perpendicular KM, and that is the Mean required. For IK : KM :: KM : KL (by 645 and 660.)

PROBLEM XXI.

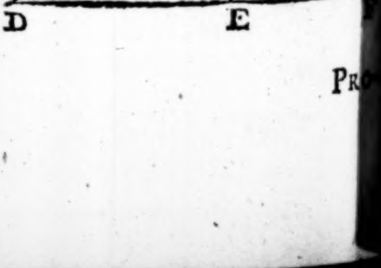
697. *Two Right-lines* A *and* B, *being given to find a third proportional* C.



Make at Pleaſure the Angle DEF; on EF take EG = A, and on ED make EH = B; and make GF = B, alſo join HG; then draw FI parallel to GH; and the Line required is HI. For EG : GF : (= EH) :: EH : HI (by 656.)

PROBLEM XXII.

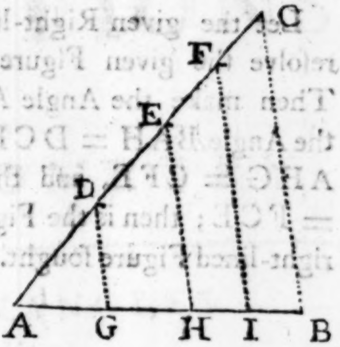
698. *Three Right-lines being given* DE, EF, DG, *to find* *fourth proportional* GH.



Join EG, and thro' F draw FH parallel to EG, and meeting DG produced in H; then is GH the Line required. For DE : EF :: DG : GH.

PROBLEM XXIII.

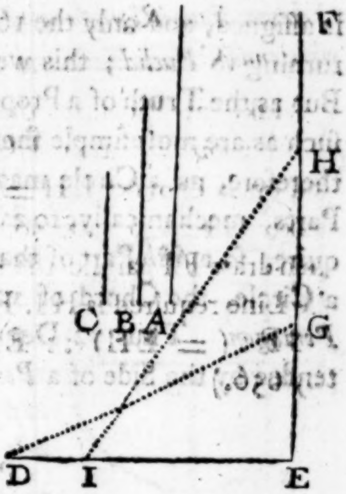
699. *To divide a Right-line* AB *as another given Right-line* AC *is divided.*



Join the Extremities of each Line CB, and parallel to CB draw EI, EH, DG through the ſeveral Points of Diviſion in the Line AC, and they ſhall divide the Line AB as AC is divided.

PROBLEM XXIV.

700. *To make a Square equal to any Number of Squares given.*



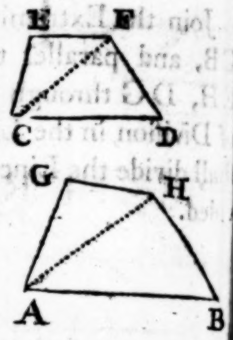
Let the Sides of three given Squares be the Right-lines A, B, C, and make the Right-angle DEF; and take ED = A, and EG = C, and join DG; then is $DG^2 = DE^2 + GE^2$ (646.) Again, make EH = DG, and EI = B; and join IH. Then is $IH^2 = IE^2 + EH^2 = EI^2 + DE^2 + GE^2 = A^2 + B^2 + C^2$. And thus you proceed for any other Number.

701. What is here done with regard to Squares, holds good for all ſimilar plain Figures deſcribed on the Sides of the Right-angled Triangle, becauſe they have all the *Ratio of the Squares* of their like Sides (by 669, 670); and how to make one Figure *ſimilar* to another is ſhewn in the following

Problem XXV.

702. *On a given Right-line to make a right-lined Figure* AGHB *ſimilar and alike ſituated to a given right-lined Figure* CEFD.

Let the given Right-line be AB, and reſolve the given Figure into Triangles. Then make the Angle ABH = D, and the Angle BAH = DCF, and the Angle AHG = CFE, and the Angle HAG = FCE; then is the Figure AGHB the right-lined Figure ſought.

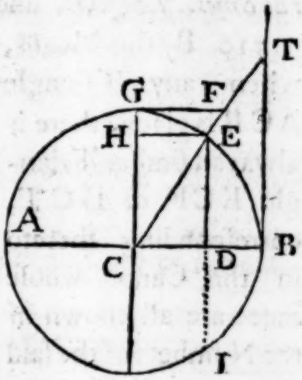


703. Theſe Problems are in general ſelf-evident to every on who underſtands the preceeding Theorems, ſo that a formal Demonſtration would have been Tautology; the Rationale of ſom is aſſigned, and only the 16th will give the Reader the Trouble o turning to *Euclid*; this we ſhall probably hereafter demonſtrat But as the Truth of a Propoſition may be ſhewn different Ways ſuch as are moſt ſimple ſhould firſt be propoſed to Learners; an therefore, as a Circle may be divided into any Number of equa Parts, mechanically, to as great Exactneſs as any Praxis can require, ſo a *fifth* Part of that Number will aſſign an Arch of ſuc a Circle, the Chord of which will be the Side of an inſcribe *Pentagon.* Thus 72 Degrees is the 5th Part of 360, and is ſubtended by the Side of a *Pentagon.*

INSTITUTIONS
OF
Plain TRIGONOMETRY.

Of RIGHT-ANGLED TRIANGLES.

704. BEFORE we can treat of the *Method of meaſuring Heights* and *Diſtances*, and other *Trigonometrical* Operations, ſo frequent in the *practical Part of the Mathematics*, it is neceſſary in the firſt Place, to explain the *Theory*, that the Grounds and Reaſons thereof may appear; and which we ſhall make the Subject of this Chapter.



705. Let AIBG be a Circle, whoſe Diameter is AB, and Center C. On the Point B erect the Perpendicular TB, which ſhall touch the Circle in B, and is therefore called a TANGENT Line; and draw CT cutting the Circle in E, and the Tangent in T, this is called the SECANT of the Arch BE. Through E draw EI ∥ TB, then is EI called the CHORD of the Segment EBI, or EAI; and Half thereof ED is called the SINE of the Arch EB or EGA. The Semidiameter CE in this Caſe is called the RADIUS.

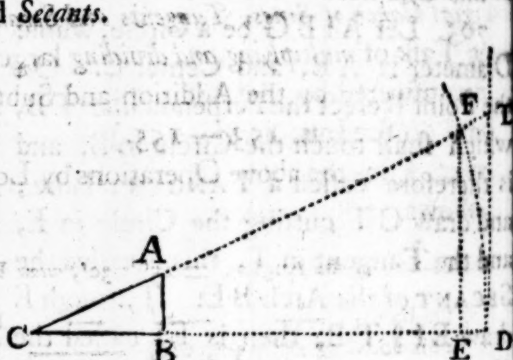
706. As every Circle is ſuppoſed to be divided into 360 equal Parts, called DEGREES, ſo one fourth Part, *viz.* 90 Degrees, is the Meaſure of the *quadrantal Arch* GB, or the Right-angle GCB. Hence whatever Number of Degrees are contained in the Arch BE, the Arch EG contains the Remainder or Complement to 90; and is therefore called the Complement of EB to a Right-angle or Quadrant BG. Hence the Tangent GF, Sine EH, and Secant CF, are ſaid to be the *Sine*, *Tangent*, *and*

Secant of the Complement EG, or in ſhort, the *Co-ſine*, *C* *tangent*, and *Co-ſecant* of the Arch BE.

707. Therefore in any Right-angled Triangle CDE, if th Side CE (ſubtending the Right-angle) which is called the *Hy pothenuſe*, be made *Radius*; then is the perpendicular Leg, o *Cathetus* ED, the *Sine* of the Angle ECB at the Baſe; an the Baſe CD is the Co-ſine thereof; for it is equal to HE, th Sine of the complemental Arch GE (706.)

708. In any Right-angled Triangle CBT, if either of th Sides be made *Radius*, as CB; then the other Side BT is th Tangent, and the Hypothenuſe CT is the Secant of the An-gle at C.

709. In Calculations of this Kind, the Sides of the Triangl are eſtimated in Parts, of which the Radius contains 10000000 and by the Induſtry of our Predeceſſors, the Sides of a Triangle whether *Sines*, *Tangents*, or *Secants*, are computed in this Mea-ſure of equal Parts for every Magnitude of the Angle BCG that is, for every Degree and Minute of the Quadrant BG; an theſe Numbers diſpoſed in Tables, are called the *Canon* of *natu-ral Sines*, *Tangents*, and *Secants*.



710. By this Means, when any Triangle ACB is given, there is always a ſimilar Triangle ECF or DCT, correſponding thereto in the Canon whoſe Sides are all known in the Numbers of the ſaid Canon; with theſe the Sides of the given Triangle are compar-ed, and by that Means, thoſe which are unknown become known.

711. Thus, ſuppoſe in the given Triangle ACB, you have found the Angle ACB = 25° : 30′, and the Side CB = 356, to find the Side AC and AB. Then the Triangle in the Ta-ble, equiangular to the given Triangle, is FCE, if AC be made Radius. For in this Caſe, the tabular Radius CF = 10000000, the Sine of 25° 30′ = FCD, is FE = 4305111; and the Co-ſine, or Baſe CE = 9025853. Hence, becauſe of

ſimilar Triangles FEC, ABC, we have the following Analogies (657.)

$$\text{As}\left\{\begin{matrix}EC & : BC\\ 9025853 & : 356\end{matrix}\right\}\begin{matrix}\left\{\begin{matrix} :: FE & : AB\\ :: 4305111 & : 169{,}8.\end{matrix}\right.\\ \left\{\begin{matrix} :: FC & : AC\\ :: 10000000 & : 394{,}4.\end{matrix}\right.\end{matrix}$$

712. But if the Baſe CB be made Radius, then we have the tabular Triangle DCT; in which the Baſe CD = 10000000; the tabular *Tangent* DT = 4769755, and *Secant* CT = 11079285, by which we find AB and AC, as follows.

$$\text{As}\left\{\begin{matrix}CD & : CB\\ 10000000 & : 356\end{matrix}\right\}\begin{matrix}\left\{\begin{matrix} :: DT & : AB\\ :: 4769755 & : 169{,}8.\end{matrix}\right.\\ \left\{\begin{matrix} :: CT & : AC\\ :: 11079285 & : 394{,}4.\end{matrix}\right.\end{matrix}$$

713. But ſince in theſe natural Numbers, the Operation is very laborious and tedious; their *Logarithms* have been made uſe of, and digeſted into Tables which are called the *logarithmical or artificial Canon of Sines, Tangents and Secants*. By theſe the irkſome Taſk of *multiplying and dividing* large Numbers is prevented, being anſwered by the Addition and Subtraction of their Logarithms. (See Inſt. 153 — 155.)

714. Thus the above Operations by Logarithms are very eaſy, as follows:

		Logarithms.
As the Co-ſine of the Angle C = 25° : 30′, *viz.*	EC = 9025853 =	9,9554882
is to the Baſe —— —— ——	BC = 356 =	2,5514500
So is the Sine of C = 25° : 30′, *viz.* ——	EF = 4305111 =	9,6339844
		12,1854344
To the Side or Cathetus —— ——	AB = 169,8 =	2,2299462

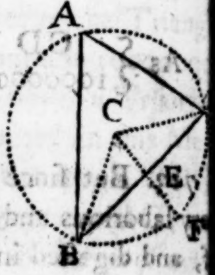
715. Or by Tangents thus,

As Radius —— ——	CD = 10000000 =	10,0000000
to the Baſe —— ——	BC = 356 =	2,5514500
So is the Tab. Log. Tangent of 25° : 30′ =	DT = 4769755 =	9,6784961
to the Side —— ——	AB = 169,3 =	2,2299461

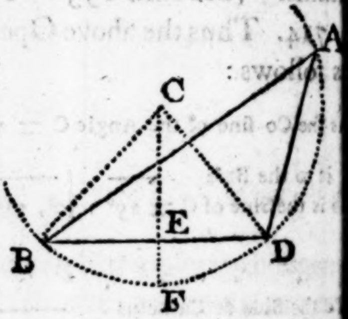
Here the Reader will obſerve the Side AB is found much eaſier by the Tangent than by the Sine, becauſe of the Radius, which gives no Trouble in the Work. Therefore in Calcu tions we ſhould always chuſe that Analogy in which the Rad makes one Term.

716. In a Right-angled Triangle, if one Angle be known, t other is known alſo, as being the Complement to 20 Degr (706.) Therefore if any one Side and one Angle be given, t other two Sides may be found; or if two Sides be given, t other Side and the Angles may be found in the Manner abo exemplified, of which we ſhall have Inſtances enough her after.

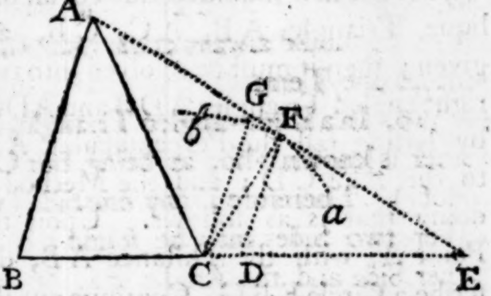
OBLIQUE ANGLED TRIANGLES.



717. About the Acute angled Triangle ADB, circumſcribe a Circle (695,) from whoſe Center C let fall the Perpendicular CE to the Side DB, which it will biſect in E, and the Arch BD in F. Join CD and CB, then is the Angle DCB = 2 DAB (642) = 2 FCB; therefore the Angle DAB = FCB; but BE is the Sine of the Angle FCB, and therefore alſo of the Angle DAB. Th $\frac{1}{2}$AB is the Sine of the Angle ADB; and $\frac{1}{2}$AD the Sine ABD.

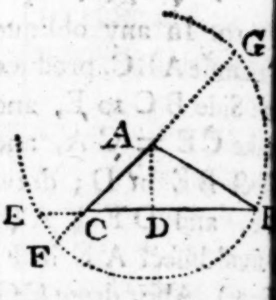


718. The ſame Demonſtration holds for any Obtuſe angled Triangle as is evident in the annexed Figure; and therefore ſince in every oblique Triangle, the Half-ſides are as the Sines of the oppoſite Angles, the whole Sides will have the ſame Ratios. And hence, if two Sides AB, AD, and an oppoſite Angle B be given, the other Parts of the Triangle will be ther by found. For as AD : Sine of ABD : : AB : Sine of A which is therefore known. Then as the Sine of B : AD : Sine of A : BD; and ſo the whole Triangle is known.



719. In any oblique riangle ABC, produce e Side BC to E, and ake CE = CA, and ſect BE in D; draw E, and DF ∥ AB, ſhall biſect AE in F, 99.) Alſo, draw CG AB. On the Center C, with the Radius CF, deſcribe the rch *a*F*b*; then becauſe the Angle ACE = B + BAC (= ACF) by (632.) and the Angle GCA = A; if therefore m the half Sum $\frac{A+B}{2}$ = ACF you take the leſſer Angle = ACG, there will remain ½ their Difference $\frac{B-A}{2}$ = CF. Alſo, if from the ½ Sum of the Sides $\frac{BC+AC}{2}$ = D, you take the leſſer Side BC, there will remain ½ their fference $\frac{AC-BC}{2}$ = CD (221, 228.) Now AF is the ngent of the Angle ACF, and GF is the Tangent of GCF 05.) and it is BD : CD :: AF : GF (656.) *Therefore,* ½ *the Sum of the Sides* $\frac{AC+BC}{2}$ *is to half their Difference* $\frac{C-BC}{2}$, *ſo is the Tangent of half the Sum of the oppoſite Angles* $\frac{+B}{2}$ *to the Tangent of half their Difference* $\frac{B-A}{2}$.

720. By this Theorem when any two Sides AC, BC, of a riangle are given, with the included Angle C; the other Side d Angles become known. For ſince the Sum of the two An- s A + B = 180 − C (633.) it is a known Quantity; and ing found ½ their Difference by the Analogy (719.) by add- it to ½ the Sum, it gives the greater Angle B, (221.) and ſubtracting it from ½ the Sum you have the leſſer Angle A. d having found all the Angles, the Side AB is known (718).

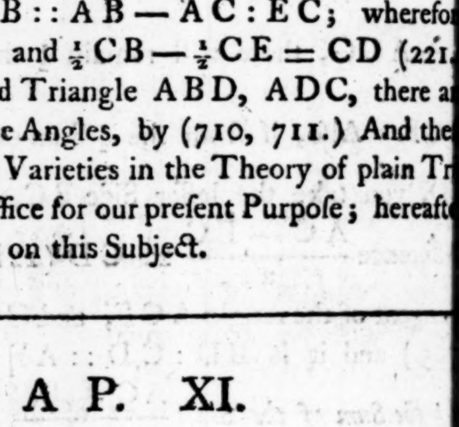


721. When the three Sides of an oblique Triangle AB, AC, CB, are given; then it muſt be reſolved into two right angled Triangles ADB and ADC, by letting fall the Perpendicular AD to the Baſe CD; and the Method of doing that is as follows. Upon the Point A, with the Diſtance AB, deſcribe a Circle EBG. Continue out the Side AC, both Ways, t cut the Circle in F and G, and the Side BC to cut it in E; the is $GC = AC + BA$, and $DE = DB$; therefore, $EC =$ $BD - CD$; thus alſo $FC = AB - AC$. But it is FC $CG = EC + CB$ (658.) whence $CG : CB :: CE : CF$ that is, $AC + AB : CB :: AB - AC : EC$; wherefo $\frac{1}{2}CB + \frac{1}{2}CE = DB$, and $\frac{1}{2}CB - \frac{1}{2}CE = CD$ (221. Then in each right-angled Triangle ABD, ADC, there a two Sides given, to find the Angles, by (710, 711.) And the are all the Particulars and Varieties in the Theory of plain Tr gonometry, which will ſuffice for our preſent Purpoſe; hereaft we may treat more largely on this Subject.

CHAP. XI.

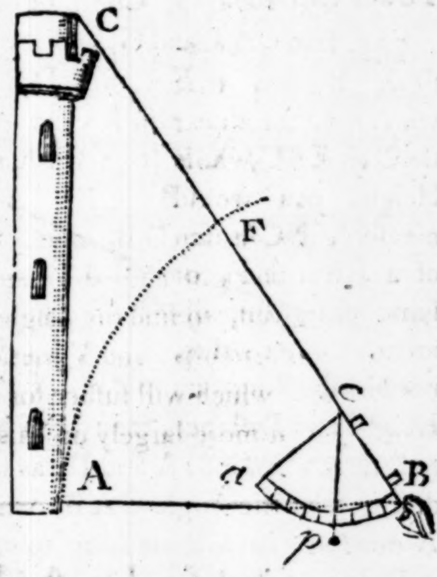
Of meaſuring HEIGHTS, DEPTHS, *and* DISTANCE

722. THIS Kind of Menſuration depends upon the Uſe of the QUADRANT in taking Angles. The Manner of doing which is therefore firſt to be explained. The *Quadrant* is ſo called from its being a *fourth Part of a Circle*, as ACB, whoſe Limb AB is accurately divided into 90



Degre

Degrees, and when the *Quadrant* is large, each Degree is ſubdivided into 10 equal Parts, each containing ſix Minutes; in the Center C is fixed a ſmall Pin, from which there hangs a Plumb-line, CP, whoſe Plummet P keeps it tightly ſtretch'd over the Degrees on the Limb when in Uſe. Upon the Side AC, are two thin Plates of Braſs, *a b*, with ſmall Perforations, or Holes, exactly in a right Line *a b*, parallel to the Side of the *Quadrant*, AC.



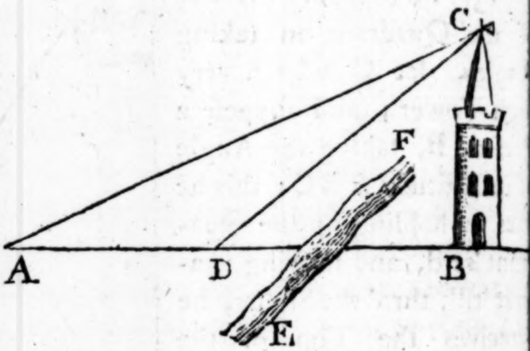
723. To illuſtrate the Uſe of the Quadrant in taking Angles, let C A be a very high Tower; and ſuppoſe a Perſon, B, taking the Angle of its Altitude ABC; this he does by holding up the Quadrant *ac*B, and moving it about 'till, thro' the Sights, he perceives the Top of the Tower C; then will the Plumb-line *cp* cut the Limb in that Number of Degrees and Minutes which are equal to the Angle of Altitude ABC, or ſo as to make the Arch *ap* of the ſame Number of Degrees as are contained in AF, deſcribed on the Radius AB. For ſuppoſing AB parallel to the Horizon, *cp* parallel thereto (as being perpendicular to AB,) and BC the viſual Ray by which the Point C appears thro' the Sights. Then the Angle $BAC = (B + C =)\ acB = acp + pcB$; but $pcB = ACB$ (631.) therefore $acp = CBA$. *Q.E.D.*

724. Hence the Height of the Tower DC is eaſily found thus; meaſure the Diſtance ED upon the Ground with a Chain or Pole, and ſuppoſe it be found 150 Yards. Then with your Quadrant at E take the Angle of Altitude ABC, which ſuppoſe you find 52° 30′. Then in the right angled Triangle ABC, there is known the Baſe $AB = DE = 150$, and the Angle at Baſe $ABC = 52° 30′$, to find the Cathetus AC. Which you do by the following Analogy (715.)

As Radius —— —— 10.
to the Baſe, —— AB = 150 = 2.176091
So is the Tangent of ABC = 52° 30′ = 10.115019

to the Height —— AC = 195,5 = 2.291110

Let the Height of the Obſerver B = 6 Feet, or 2 Yards; the AC + AD (or B) = DC = 197½ Yards, the Height of th Tower required.



725. It may ſometimes happen that you cannot come near the Object BC, whoſe Height you would meaſure, by Reaſon of a River EF, or ſome other Impediment. Yet this, notwithſtanding its Height and Diſtance, may both be meaſured by taking its Alt tude at two Stations A and D, as follows. At A, with the Qua drant, take the Angle CAB, which let be 24° 45′; then fro A meaſure, in a ſtrait Line towards the Object, the Diſtanc AD = 100 Rods; and at D take the Angle CDB, which ſu poſe = 38°. Then is the Angle ADC = 180 — 38 = 142 So that in the obtuſe Triangle ADC, there is known the Sid AD, and the adjacent Angles A and D, to find the Side DC which is done thus. The Angle ACD, CDB — CAB = 38° — 24° 45′ = 13° 15′. Then ſay, (718.)

As the Sine of the Angle ACD = 13° 15′ = 9.36021

is to the Side —— AD = 100 = 2.00000
So is the Sine of the Angle DAC = 24° 45′ = 9.62186

to the Length of the Side DC = 182,6 = 2.26164

726. Then in the right angled Triangle DBC, there is give the Hypothenuſe DC, and the Angle at Baſe, to find the Sid CB, and DB. Therefore ſay,

As Radius — — — 10.
to the Side — — BC = 182,6 = 2.261646
So is the Sine of the Angle CDB = 38° 00′ = 9.789342

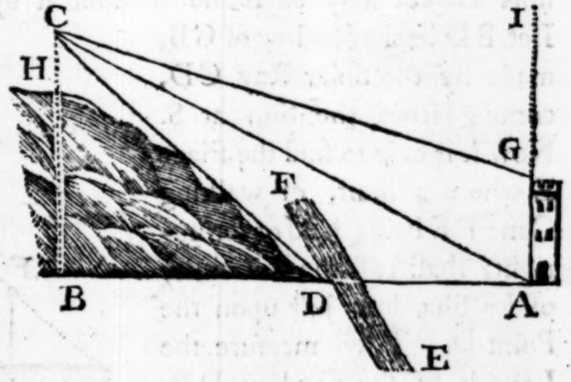
to the Height of the Spire CB = 112,46 = 2.050988

727. Then to find the Diſtance DB,

Say, as Radius — — 10.
to the Side — BC = 182,6 = 2.261646
So is the Sine of the Angle DCB = 52° 00′ = 9.896532

to the Diſtance — DB = 143,9 = 2.158178

728. In the ſame Manner, if you were ſtationed on the Top of an high Hill, as at H, and at the Foot of the Hill a River EF, or ſome other Obſtacle prevented your Acceſs to the diſtant Object at A; yet if the Side of the Hill HD was not too irregular, you might eaſily find the Height of the Hill, and the Diſtance and Height of the Object AG. Thus, let the ſlant Height, or Slope of the Hill HD be meaſured, and ſuppoſe it is found 182,6 Rods. Then will this differ inſenſibly from the Hypothenuſe CD of the right angled Triangle BCD; with the Quadrant you take either of the Angles BCD, or BDC; thus let BDC = 38°, then will BCD = 52°. And having the Hypothenuſe CD = 182,6 and Angle BDC = 38°. You find the perpendicular Height of the Hill (or Eye at C) to be CB = 112,46; and the Baſe of the Hill BD = 143.9 (by 727.)

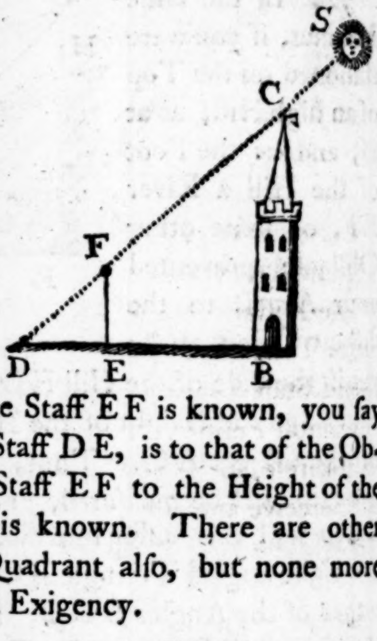


729. Having thus got the Height of the Hill, take with your Quadrant the Angle BCA, which you will find to be 65° 15′; and becauſe the Angle BDC = 38°, the Angle CDA = 142°, and the Angle DCA = 65° 15′ — 52° = 13° 15′. Therefore in the obtuſe Triangle DCA, there is given all the Angles, and the Side CD, by which Means the Side DA will be found 100

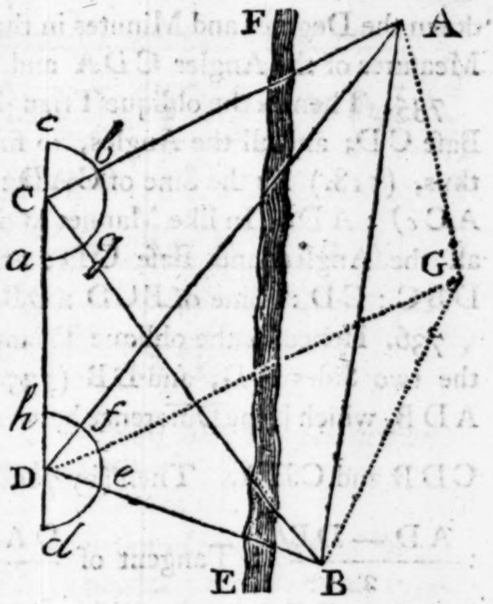
Rods (725.) Which is the Diſtance of the Object from the Foo of the Hill required.

730. Moreover if it be required to meaſure the Height of diſtant Object A G; then you next find the Side CA of th Triangle ADC; the Angle BCG being taken with the Quadrant, is equal to the Angle CGI; whence the Angle CGA is known. Alſo the Angles $CAG = ACB = 65^\circ\ 15'$; hence all the Angles, and the Side AC in the Triangle AGC are known. Then ſay, as the Sine of CGA (or IGC, (705.) is to the Side AC :: the Sine of ACG to the Side AG = the Height of the Object required.

731. If a Quadrant be not at Hand, the Height of any acceſſible Object may be found without it by the Shadow, thus. Let BD be the Shadow of CB, made by the Solar Ray CD, coming from the Sun at S. Now it is eaſy to find the Place E where a Staff, or walking Cane EF being held or ſet upright, ſhall caſt the Extremity of its Shaddow juſt upon the Point D. Then meaſure the Length of the Shadow DB, and DE, and as the Height of the Staff EF is known, you ſay (by 657.) As the Shadow of the Staff DE, is to that of the Object DB, ſo is the Height of the Staff EF to the Height of the Object BC, which, therefore is known. There are other Ways of doing this without a Quadrant alſo, but none more eaſy, and ſo ready as this upon an Exigency.



N. B. As the meaſuring the *Depth* of a Valley, or other Place, depends on the ſame Principles and Conſtruction of Figures, I apprehend it will not be neceſſary to inſiſt on that Article here, as there can be no Difficulty attending it, if Inſtitution 728 be at all underſtood; ſince the *Height of the Hill* is the ſame Thing with the *Depth of the Vale*.



732. To meaſure the Diſtance between two or more inacceſſible Objects A, B, you muſt be provided with a Semi-circle, with a moveable Index carrying *two Sights*, and a Spirit-Level for fixing it truely horizontal; a *Plane-Table* alſo will anſwer the End. And a *Theodolite* much better than any other Inſtrument. Some Inſtrument of this Sort muſt be had, and it matters not much for the Form; we will therefore ſuppoſe it the common graduated *Semi-circle*.

733. In order, then, to meaſure the Diſtance A, B, on the other Side of the River E F, two Stations muſt be determined, as C and D, in which to fix the Semi-circle for taking Angles; let the Diſtance C D be nicely meaſured, for the Baſe of your future Triangles. Then over the Point C erect your Three-legged Staff bearing the Semi-circle, and adjuſting it to a true horizontal Poſition, move the Index to the Beginning of the Semi-circle, or Line C*a*, and move it about 'till thro' the Sights you perceive an upright Pole, *&c.* fixed before-hand in the Point D. Here let the Semi-circle reſt; and then move the Index 'till thro' the Sights at C*g* you ſee the Object B, and write down the Number of Degrees and Minutes contained in the Arch of the Semi-circle *ag*, for that will be the Meaſure of the Angle DCB. Again, move the Index to C*b*, where you obſerve the Object A, and write down the Degrees and Minutes in the Arch *ab*, which is the Meaſure of the Angle D C A.

734. This done, remove your Semi-circle to D, and there ſetting it up, with its Centre over D, lay the Index to the Side D*b*, and move the Semi-circle about 'till thro' the Sights you obſerve the upright Pole, *&c.* in C. In this Poſition ſcrew the Semi-circle faſt, and move the Index to D*f* and D*e*, and write

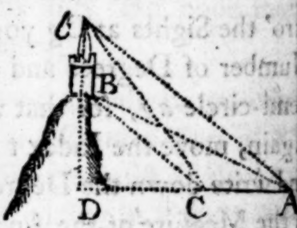
down the Degrees and Minutes in the Arches *bf* and *be*, for t Meaſures of the Angles CDA and CDB.

735. Then in the oblique Triangle CAD, there is given t Baſe CD, and all the Angles, to find the Side AD; which thus, (718.) As the Sine of CAD : CD :: Sine of ACD (AC*c*) : AD. In like Manner in the Triangle CBD you hav all the Angles and Baſe CD; and therefore, as the Sine DBC : CD :: Sine of BCD : DB.

736. Hence in the oblique Triangle ADB, there is know the two Sides AD, and DB (735.) and the included Ang ADB, which is the Difference between the known Angles (734. CDB and CDA. Then (by 719.) you ſay, as $\frac{AD + D}{2}$: $\frac{AD - DB}{2}$ :: Tangent of $\frac{DAB + DBA}{2}$: Tangent $\frac{DAB - DBA}{2}$. Then, as you have ½ the Sum, and ½ th Difference, of thoſe Angles, the Angles themſelves are know (by 720.)

737. Laſtly, In the Triangle ADB, all the Angles, and tw Sides AD and DB are known to find the third Side AB, thus, (718. as the Sine DAB : DB :: Sine of ADB : AB = the Diſtanc between the two Objects A and B as required. In the ſam Manner the Diſtance between A and G, or B and G may b found.; and thus the moſt uſeful Doctrine of plain Triangles ex tends to *Surveying*, *Fortification*, *Navigation*, and every othe Art where right-lined Figures are concerned.

738. The ſame Method is taken for meaſuring the Height of an Object B*b*, on the Top of an inacceſſible Mountain BD, as is evident by a bare Inſpection of the Figure, the Conſtruction here being the ſame as the Foregoing, and therefore the Particulars need not be repeated.



INSTITUTIONS
OF
CONIC GEOMETRY.

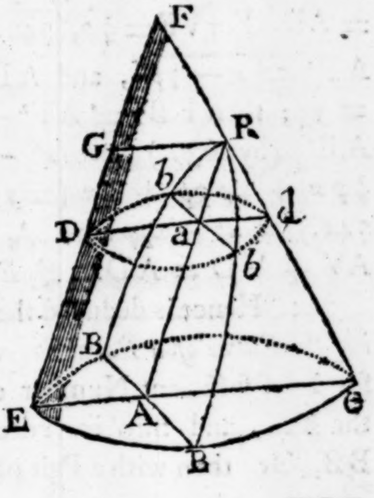
CHAP. XII.

The Chief Properties *of the* CONIC SECTIONS *demonſtrated.*

39. BEFORE we can proceed to the Menſuration of *Superficies* and *Solids*, it will be neceſſary to premiſe the Prin- iples of CONIC GEOMETRY, and the Method of FLUXIONS; s by theſe we ſhall come to the Rationale of thoſe uſeful Prac- ces, ſooner and eaſier than by any other Way. Nor is the Rea- er to wonder, if he be told, than he can never guage a Veſſel, r meaſure a Piece of Timber, *according to Art*, 'till he has firſt arned *Conics* and *Fluxions*, or obtained the ſame Ideas ſome ther Way, which he will find much more difficult, laborious, d irkſome. I have been at the Pains to try every Way myſelf, d therefore am able to point out, to the young *Tyro*, that hich is pleaſanteſt and beſt.

Of the PARABOLA.

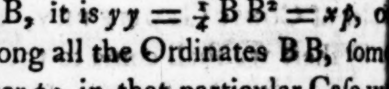
740. If a Cone be cut thro' by Plane parallel to one of its Sides F, the Figure of the Section A RB is called a PARABOLA; d its Property is thus inveſtiga- d. EB*e*B is a Circle, E*e* a ameter, and BB (the Baſe of e Parabola) a right Line at right ngles therewith. AR is the Ax- of the Parabola; draw GR pa- llel to E*e*. Put FG $= a$, EA $= b$, AR $= x$, and AB $= y$. hen, becauſe of ſim. Triangles,



FGR and RAe, it is FG : GR (= EA) :: AR : Ae,
that is, $a : b :: x : \frac{bx}{a} = Ae$. But EA × A$e$ = AB² (b
658.) That is $\frac{b^2 x}{a} = yy$. Now put $\frac{b^2}{a} = p$, then $px = yy$
After the ſame Manner it is ſhewn, that, (drawing Dd ∥ Ee
and bb ∥ BB, and putting Ra = x, and ab = y) it is px =
yy, therefore $p = \frac{yy}{x} = \frac{\mathrm{yy}}{\mathrm{x}}$; whence y² : y^2 :: x : x. O
Ra : RA :: ab² : AB², every where.

741. The Part of the Axis Ra, or RA, is called the *Ab-ſciſſa*, the Lines ab, AB *Ordinates*, or Lines ordinately applie to the Axis. The invariable Line p is called the *Parameter*, o *Latus Rectum*, being a third Proportional to any Abſciſſa and it Ordinate; for ſince $px = yy$, we have $x : y :: y : p$.

742. Since y = AB = ½ BB, it is yy = ¼ BB² = xp, o 4px = BB × BB. Now among all the Ordinates BB, ſom one will be equal to the Parameter p; in that particular Caſe w have $4px = pp$, or $4x = p$, and ſo $x = \frac{1}{4}p$; that Point of th Axis, where this Parameter-ordinate interſects it, is called th Focus *of the Parabola*. As the Point F, when the Ordina PR = 4 VF, the *Abſciſſa*. See the following Figure.

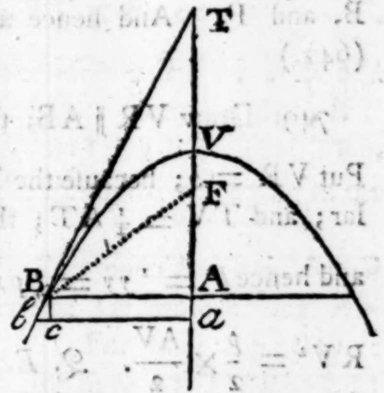


743. In the Axis continued, take VD = VF, the Diſtance of the Focus from the Vertex, and join FB; then it will be FB = AV + VF = AD. For VF = ¼p, (742.) AF² = $\overline{x - \frac{1}{4}p}^2$, and AB² = yy; then FB² = AF² + AB² (by 636.) = $x^2 - \frac{1}{2}px + \frac{1}{16}pp + px$ (= yy, 741.) = $x^2 + \frac{1}{2}px + \frac{1}{16}pp$; therefore FB = $x + \frac{1}{4}p$ = AV + VD = AD. *Q.E.D.*

744. Hence is deduced the beſt and eaſieſt Method of *deſcri-ing the Curve of a Parabola*, whoſe Focal Diſtance VF is give for let a ſufficient Number of Points A, A, &c. be taken i the Axis, and thro' each draw perpendicular and parallel Lin BB, &c. then with a Pair of Compaſſes take the Diſtance A

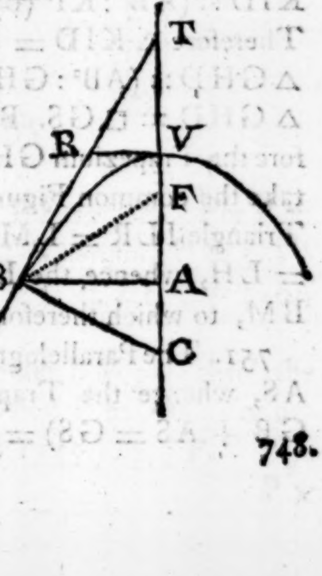
an

and with one Foot in the Focus F, interſect each Parallel reſpectively, in B, B; then with an even Hand draw a Curve through all thoſe Points, and it ſhall be the *Parabola* propoſed.



745. If any Right-line touch the Parabola as in B, the ſame ſhall meet the Axis produced in T, ſo as to make AV = TV. For let B*b* be an indefinitely ſmall Part of the Curve coincident nearly with the Tangent TB. Draw $ba \parallel$ BA, and $Bc \parallel Va$; and put $Bc = n$, $bc = m$, and $TV = a$. Then $Va = x + n$, and $ab = y + m$; and in the Triangles bBc, BTA, we have $m : n :: y : x + a$, therefore $n\frac{y}{m} = x + a$; but $p \times Va = \overline{ab}^2$, that is, $px + pn = y^2 + 2ym + m^2$, but $px = yy$; therefore $pn + yy = y^2 + 2ym$ (for m^2 is too ſmall to be regarded) hence $pn = 2yn$ and $n = \frac{2ym}{p} = \frac{xm + ma}{y}$, or $\frac{2y}{p} = \frac{x + a}{y}$, that is, $2yy = px + pa$, and $p = \frac{2yy}{x + a} = \frac{yy}{x}$, conſequently $2x = x + a$, therefore $x = a$, or AV = TV.

746. From the Focus F draw FB, it ſhall be FB = FT, for AT $= 2x$ (745.) and AF $= x - \frac{1}{4}p$; therefore FT = AT − AF $= 2x - x + \frac{1}{4}p$ = FB (by 743).



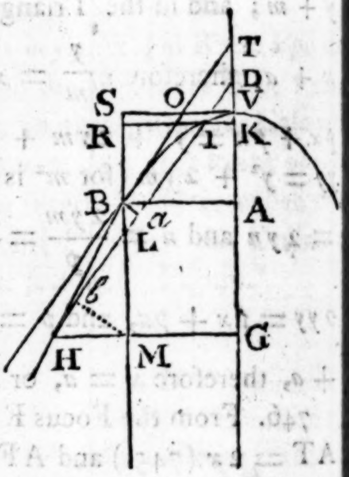
747. To the Tangent TB, let CB be Perpendicular in the Point B, and meet the Axis in C; then it will be AC $= \frac{1}{2}p$. For let AC $= b$, then (by 660.) it is AT : AB :: AB : AC, that is, $2x : y :: y : b = \frac{yy}{2x} = \frac{1}{2}p$, (by 741.) = AC.

748. Moreover it is, $FB = FC$; for $AF = x - \frac{1}{4}p$, and $AC = \frac{1}{2}p$ (by 747.) therefore $FC = x - \frac{1}{4}p + \frac{1}{2}p = x + \frac{1}{4}p = BF$ (746.) $= FT$.

Coroll. Hence F is the Center of a Circle paſſing through C, B, and T. And hence alſo the Angle $CTB = \frac{1}{2}CFB$. (642.)

749. Draw $VR \parallel AB$, then it ſhall be $VR^2 = \frac{p \times AV}{4}$. Put $VR = b$; becauſe the Triangles TVR, TAB, are ſimilar; and $TV = \frac{1}{2}AT$; therefore $RV = \frac{1}{2}AB$, or $b = \frac{1}{2}y$; and hence $b^2 = \frac{1}{4}yy = \frac{1}{4}px$ (741.) $= \frac{p}{2} \times \frac{x}{2}$; therefore $b^2 =$

$$RV^2 = \frac{p}{2} \times \frac{AV}{2}.$$ *Q. E. D.*



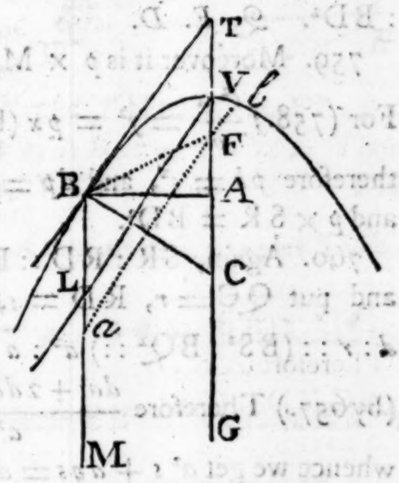
750. Draw $DH \parallel TB$ a Tangent, cutting the Curve in I and H; and from the Point B, draw $BM \parallel VG$, this is called *a Diameter to the Point* B; and it ſhall biſect the Part IH within the Curve. For through the Points V, I, H, draw VS, KR, GH, parallel to AB, and meeting BM produced in S, R, M. Then becauſe $VO = \frac{1}{2}AB = \frac{1}{2}VS$, and $TV = SB$, (745.) therefore the Triangle $TVO = SBO$, and $\triangle ABT = \square AS$. But $\triangle ABT$ (or AS) : KID :: ($AB^2 : KI^2$ (670.) :: AV : KV (740.) AS : KS (655). Therefore $\triangle KID = \square KS$. Again, $\triangle ABT$ ($= \square AS$) : $\triangle GHD$ :: ($AB^2 : GH^2$ :: AV : GV) $\square AS : \square GS$; therefore $\triangle GHD = \square GS$. But $GHD - KID = GS - KS$; therefore the Trapezium $GHIK = \square GR$. From each of which, take the common Figure GMLIK, and there will remain the Triangle $ILR = LMH$; and as they are ſimilar, we have $LI = LH$, whence the Line IH is biſected in L by the Diameter BM, to which therefore the Line HI is an *Ordinate*.

751. The Parallelogram $LBTD = \triangle MLH$; for $ABT = AS$, whence the Trapezium GMBT ($= GB + ABT = GB + AS = GS$) $=$ (750.) GHD; from which take the

common Figure HMLD there remains $LBTD = \triangle LMH$. *Q. E. D.*

752. From B let fall the Perpendicular Ba; and from M the Perpendicular Mb. Then becauſe $BT \times Ba = \frac{LH \times Mb}{2}$, or $2BT \times Ba = LH \times Mb$; therefore $2BT : LH :: Mb : Ba :: LM : LB$, by ſimilar Triangles. Conſequently $2BT \times LB = LM \times LH$. Let $BS : BO :: 2BT : P$. Then ſhall this fourth proportional P be the *Parameter* belonging to the Diameter BM.

753. Then, the Rectangle of this Parameter and any *Abſciſſa* of its Diameter is equal to the Square of Ordinate of that Abſciſſa. Or $P \times BL = LH^2$ or LI^2. For ſince $BS : BO :: LM : LH$, by ſimilar Triangles; we have $2BT : P :: LM : LH$; whence $P = \frac{2BT \times LH}{LM}$, and (by 752.) it is $LB = \frac{LM \times LH}{2BT}$. Therefore $P \times LB = \frac{2BT \times LH}{LM} \times \frac{LM \times LH}{2BT} = LH^2$. *Q. E. D.*



754. The Parameter P is equal to $p + 4AV$. For draw VL parallel to the Tangent TB, this will be an Ordinate to the Diameter BM (by 750). Then becauſe $BM \parallel VG$, it is $TV = BL = AV$ (745.) $= x$; and (by 753.) $P \times BL$, or $P \times x = VL^2 = BT^2 = AB^2 + AT^2 = y^2 + 4x^2$, therefore $Px = 4x^2 + px$ (741.); whence $P = 4x + p = p + 4AV$. *Q. E. D.*

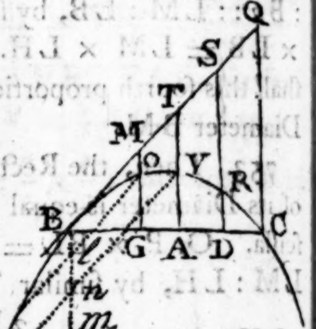
755. Let F be the Focus; and then $BF = \frac{1}{4}P$. For $P = p + 4AV$ (754.) and $p = 4FV$ (742). Therefore $P = 4AV + 4FV$, and $\frac{1}{4}P = AV + FV = FT = BF$ (by 746.) *Q. E. D.*

756. If an Ordinate ab to any Diameter BM paſs thro' the Focus F, then it ſhall be $\frac{1}{4}P^2 = ab^2$, or $\frac{1}{2}P = ab$. For becauſe of Parallels, $aB = FT = BF$ (755.) $= \frac{1}{4}P$. Alſo

P ×

$P \times ab = (753.)\ \overline{ab}^2$; therefore $\frac{1}{4}P \times P = \overline{ab}^2$; whence $\frac{1}{2}P = ab$. *Q. E. D.*

757. Let CB be perpendicular to the Tangent in B; then is $CT = \frac{1}{2}P$. For $BF = \frac{1}{4}P$ (755.) $= CF = FT$ (748). Conſequently $CT = CF + FT = \frac{1}{4}P + \frac{1}{4}P = \frac{1}{2}P$. *Q. E. D.*



758. If a Right-line QB touch the Parabola in B, and from the Points M, S, in the Tangent, the Right-lines MG, SD are drawn parallel to the Axis VA, cutting the Ordinate BC in G and D; then it will be $MO : SR :: BG^2 : BD^2$. Put $MO = b$, $SR = d$, $BG = c$, $BD = a$; alſo let AV ($=$ VT) $= x$; then drawing $Bm \parallel VA$, and Ol, $Vn \parallel BQ$; it is $x : b :: (Bn : Bl :: nV^2 : Ol^2 ::)\ BT^2 : BM^2 :: y^2\ (= AB^2) : c^2\ (= BG^2)$ (by 657 and 671). For the ſame Reaſon $x : d :: y^2 : a^2$. Therefore (by 652.) we have $b : d :: c^2 : a^2$, or $MO : SR :: BG^2 : BD^2$. *Q. E. D.*

759. Moreover it is $p \times MO = BG^2$, or $p \times SR = BD^2$. For (758.) $\frac{c^2 x}{b} = y^2 = px$ (by 741). Alſo $\frac{a^2 x}{d} = y^2 = px$; therefore $pb = c^2$, and $dp = a^2$; that is, $p \times MO = BG^2$, and $p \times SR = BD^2$.

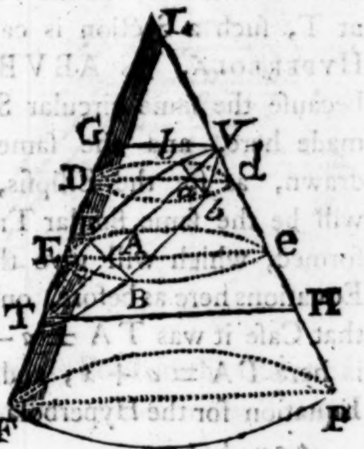
760. Again, $SR : RD :: BD : DC$; for draw $QC \parallel VA$, and put $QC = r$, $RD = s$, and $DC = q$, $R = p$. Then $d : r :: (BS^2 : BQ^2 ::)\ a^2 : \overline{a+q}^2$, and $r : a+q :: d+s : a$. (by 657.) Therefore $\frac{da^2 + 2daq + dq^2}{a^2} = r = \frac{ad + as + qd + qs}{a}$; whence we get $a^2 s + aqs = daq + dq^2$, and dividing by $a+q$, it is $as = dq$; hence $d : s :: a : q$, or $SR : RD :: BD : DC$. *Q. E. D.*

761. Again, $AV : AB^2 :: RD : BD \times DC$. For $x : d :: y^2 : a^2$ (758.) and $d : s :: a : q$ (760.) therefore $\frac{xa^2}{y^2} = d = \frac{sa}{q}$, hence $xqa = y^2 s$; then $x : y^2 :: s : aq$, or $AV : AB^2 :: RD : BD \times DC$. *Q. E. D.*

Coroll. Hence alſo (AV : AB² : :) OG : BG × GC : : RD : BD × DC.

762. Becauſe the $y^2 = px$ (741.) and $y^2 = \frac{aqx}{s}$ (761.) therefore $px = \frac{aqx}{s}$, and $ps = aq$; that is, $p \times$ RD = BD × DC. Thus alſo $p \times$ OG = BG × GC. *Q. E. D.*

Of the ELLIPSIS.

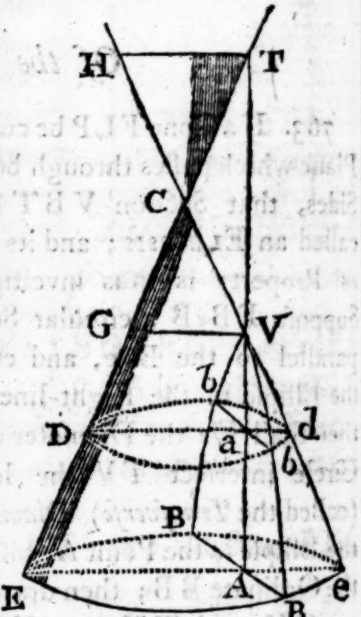


763. If a Cone FLP be cut by a Plane which paſſes through both its Sides, that Section VBTBV is called an ELLIPSIS; and its general Property is thus inveſtigated. Suppoſe EB*e*B a circular Section parallel to the Baſe, and cutting the Ellipſe in the Right-line BB; then ſhall E*e* the Diameter of the Circle interſect TV the longeſt (called the *Tranſverſe*) *Diameter* of the Ellipſe in the Point A, biſecting the Ordinate BB; then draw GV and TH parallel to FP; and put GV = d, TH = c, VT = a, AB = y, VA = x. Then becauſe of ſimilar Triangles VA*e*, VTH, we have VT : TH : : VA : A*e* $= \frac{cx}{a}$. Alſo becauſe of the ſimilar Triangles TGV, TEA, it is, TV : GV : : TA : AE $= \frac{\overline{a-x} \times d}{a} = \frac{ad - xd}{a}$. But becauſe of the Circle EB*e*B, we have EA × A*e* = AB × AB = (658.) AB²; that is, $\frac{cxad - cxxd}{aa} = yy$.

764. Let it be made $a : d : : c : p = \frac{cd}{a}$; then ſubſtituting p for its Value in the above Equation, it becomes $\frac{pax - pxx}{a} = yy$. And if another circular Section D*bdb*D be made cutting the Elliptic one in *bb*, then ab = y, and Va = x, it will ap-

pear by the ſame Reaſoning, that $\frac{pax - pxx}{a} = yy$. Hence $ax - xx : yy :: a : p :: ax - xx : yy$. Conſequently it will be every where $a : p :: \overline{a - x} \times x : y^2$.

Of the HYPERBOLA.



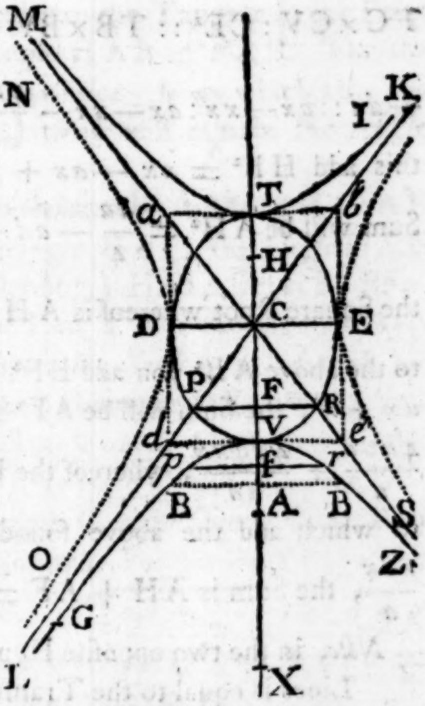
765. If a Cone E C *e* be cut by a Plane thro' the Baſe, and the oppoſite Side continued above the Vertex as at T, ſuch a Section is called an HYPERBOLA, as ABVB; and becauſe the ſame circular Sections made here, and the ſame Lines drawn, as for the Ellipſis, there will be the ſame ſimilar Triangles formed, which will give the ſame Equations here as before, only as in that Caſe it was $TA = a - x$, it is here $TA = a + x$; and ſo the Equation for the Hyperbola will be this $\frac{pax + pxx}{a} = y^2$, or in Analogy, $a : p :: \overline{a + x} \times x : yy$. Hence the Analogy $a : p :: \overline{a \mp x} \times x : yy$ will ſerve for both *Ellipſis* and *Hyperbola*, at the ſame Time; uſing the Sign — for the former, and + for the latter in all Caſes.

766. The greateſt Ordinate DE in the Ellipſe is called the *Conjugate Diameter*. Let the Half thereof of $ED = b$; in this Caſe $x = VC = \frac{1}{2}a$, and $y = \frac{1}{2}b$. Whence the Analogy for the Ellipſe becomes $a : p :: \overline{a - \frac{1}{2}a} \times \frac{1}{2}a : \frac{1}{2}b \times \frac{1}{2}b$; that is, $a : p :: \frac{1}{4}aa : \frac{1}{4}bb$, and ſo $ap = bb$; whence $a : b :: b : p$. So the *Parameter of the Ellipſe is a third Proportional to the Tranſverſe and Conjugate Diameter.*

767. Since $a : p :: aa : bb$ (766.) $:: \overline{a - x} \times x : yy$ (764.) we have $AB^2 : TA \times VA :: DE^2 : TV^2$, or $\frac{bb}{aa} \times \overline{ax - xx} = yy$.

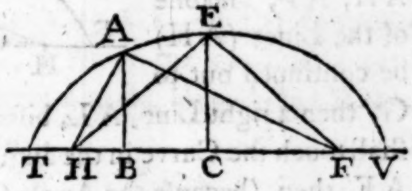
768. The

768. Becauſe, in the Focus, $y = \frac{1}{2}p$, or $yy = \frac{1}{4}pp$, we ſhall, in that Caſe, have $\frac{1}{4}p = \frac{ax - xx}{a}$, or $\frac{1}{4}ap = ax - x^2$; and changing all the Signs it is $x^2 - ax = -\frac{1}{4}ap$, and compleating the Square it is $x^2 - ax + \frac{1}{4}aa = \frac{1}{4}aa - \frac{1}{4}ap$; extract the Root, and $x - \frac{1}{2}a = \sqrt{\frac{1}{4}aa - \frac{1}{4}ap} = \frac{1}{2}\sqrt{aa - ap}$. Let F, H, be the two Focus's; then $CF = CH = \frac{1}{2}a - x = \frac{1}{2}\sqrt{aa - ap}$, or $\sqrt{aa - ap} = 2FC = FH$; whence $aa - ap = FH^2$, and ſo $TV\ (= a) : FH :: FH : TV - p\ (= a - p.)$



769. Two Right Lines drawn from any Point in the Curve of the Ellipſis to the two Foci, are, together, equal to the Tranſverſe Diameter.

CASE I.



Let E, the End of the Conj. Diameter, be the given Point; and draw HE, FE; then is $HE + FE = TV$. For $CE^2 + CH^2 = HE^2$; but $CE^2 = \frac{ap}{4}$, and $CH^2 = \overline{\frac{1}{2}a - x}|^2 = \frac{aa - ap}{4}$; whence $\frac{ap}{4} + \frac{aa - ap}{4} = \frac{aa}{4} = EH^2$; therefore $EH = \frac{1}{2}a = EF$; conſequently $EH + EF = \frac{1}{2}a + \frac{1}{2}a = a = TV$. (766, 768.)

CASE II.

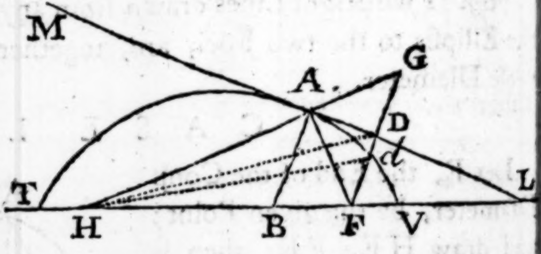
Let A be a Point taken any where in the Curve, and draw HA, FA; and the Ordinate AB. Put $TB = x$, $HC = d$; then $TH = \frac{1}{2}a - d$, $HB = x - \frac{1}{2}a + d$, and becauſe $HE = TC = \frac{1}{2}a$, therefore $CE^2 = \frac{aa}{4} - dd$. But

TC

$TC \times CV : CE^2 :: TB \times BV : AB^2$ (767.) that is $\frac{aa}{4} : \frac{aa}{4}$ $- dd :: ax - xx : ax - xx - \frac{4xdd}{a} + \frac{4ddxx}{aa} = AB^2$; to this add $HB^2 = xx - ax + \frac{1}{4}aa + 2dx - ad + dd$, the Sum will be $AH^2 = \frac{aa}{4} - ad + dd + 2dx - \frac{4xdd}{a} + \frac{4ddxx}{aa}$, the Square Root whereof is $AH = \frac{1}{2}a - d + \frac{2dx}{a}$. Alſo, if to the above AB^2 you add $BF^2 = d^2 + da + \frac{1}{4}aa - 2dx - ax + x^2$, the Sum will be $AF^2 = d^2 + da + \frac{1}{4}aa - 2dx - \frac{4xdd}{a} + \frac{4ddxx}{aa}$, whereof the Root is $d + \frac{1}{2}a - \frac{2dx}{a} = AF$; to which add the above found Equation $AH = \frac{1}{2}a - d + \frac{2dx}{a}$, the Sum is $AH + AF = \frac{1}{2}a + \frac{1}{2}a = TV$. *Q.E.D.*

Note, in the two oppoſite Hyperbolas, the Difference of theſe Lines is equal to the Tranſverſe TV.

770. If to any Point A of the Curve right Lines be drawn from the *Foci*, *viz.* AH, AF, and one of the Lines (AH) be continued out to G, then a right Line, AL, biſecting the external Angle FAG, ſhall touch the Curve in the angular Point A. For make AG = AF, then (becauſe the Angle GAL = FAL) if you take any Point, D, in the Line AL, we have GD = DF (by Inſt. 681.) draw HD, and H*d*, then HD + DG is greater than HA (+ AG =) + AF = H*d* + *d*F = TV. (769.) Therefore the Point D is without the Curve, and will be ſo 'till it coincides with the Point A, conſequently AL is a Tangent to the Curve in that Point.



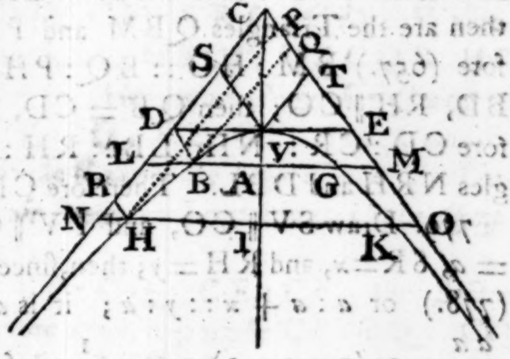
771. The two Lines AH, AF, drawn from the Point of Contact A to the two *Foci*, make equal Angles with the Tangent MAL. For the Angle FAL (= GAL (by 770.) = MAH (630.)

772. Let A B be perpendicular to the Tangent in the Point of Contact A, then ſhall the Angle H A B = F A B. For the Angle M A B = L A B, by Suppoſition; from which take the Angle M A H = L A F (771.) there will remain the Angle H A B = F A B, (200.)

773. The ſame Conſtruction remaining; we have H A : A F :: H B : B F. For in the Triangle H A F, the Angle H A B = B A F (772.) Therefore (by 666.) H A : A F :: H B : B F.

774. Since one and the ſame Line T V is the common Axis both to the *Ellipſis* and to the *Hyperbola*; (ſee Fig. to Inſt. 768.) if we alſo take one common conjugate Diameter D E to both; and about the Ellipſis deſcribe the Rectangle *abed*. Then, ſince p is the ſame in the *Ellipſis* and *Hyperbola*; we have C f = C d, or the Diſtance of the Focus f of the Hyperbola from the Center C, equal to half the Diagonal of the Rectangle *abed*. For putting V f = x, we ſhall find (by the Proceſs of 768.) $\sqrt{\frac{1}{4}aa + \frac{1}{4}ap} = \frac{1}{2}a + x$ = C f; but $ap = bb$ (by 766.) therefore $\sqrt{\frac{1}{4}aa + \frac{1}{4}bb} =$ C f $= \sqrt{CV^2 + Vd^2} =$ C d (636.) *Q. E. D.*

775. The Diagonals of this rectangular Parallelogram, produced both Way indefinitely, make thoſe right Lines which are called the **Asymptotes**, whoſe Property it is *to approach nearer and nearer to the Curve of the Hyperbola inſcribed between them, but will never coincide therewith.* Thus C S, C L are the *Aſymptotes* of the Hyperbola G V Z; C K, C M of the oppoſite Hyperbola. Alſo C M, C L are Aſymptotes to the Hyperbola O D N; and C Z, C K of the Oppoſite one.



776. That the Aſymptotes conſtantly approaches the Curve but can never meet it, is thus ſhewn. Let C V = $\frac{1}{2}a$, V D = $\frac{1}{2}b = \sqrt{\frac{ap}{4}}$ or V D$^2 = \frac{ap}{4}$; V A = x, V I = x; then C A = $\frac{a}{2} + x$, C I = $\frac{a}{2} + x$; and

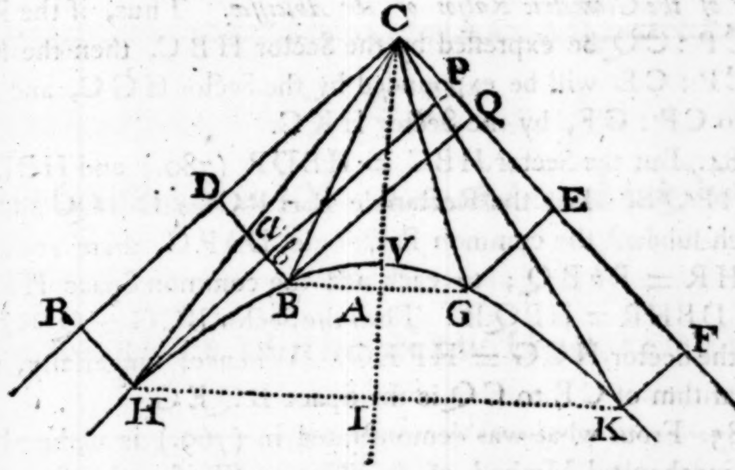
CV : VD :: CA : AL $= \sqrt{\frac{ap}{4}} + \sqrt{\frac{ap}{4}} \times \frac{2x}{a}$; and therefore $AL^2 = \frac{ap}{4} + px + \frac{pxx}{a} = \frac{ap}{4} + yy$ (765.) $= \frac{ap}{4} + AB^2$; therefore $AL^2 - AB^2 = \frac{ap}{4} = VD^2$. And by reaſoning in the ſame Manner we prove $IN^2 - IH^2 = VD^2$. Therefore, ſince the Square of the Ordinate to the Curve is every where leſs than the Square of the ſame Ordinate to the Aſymptote, by the conſtant Difference of $\frac{ap}{4}$ or VD^2, 'tis evident the Aſymptote can never touch the Curve. *Q. E. D.*

777. In the Hyperbola, it will be every where $VD \times VE = LB \times BM = NH \times HO$. For $VD \times VE = VD^2 = \frac{1}{4}ap = AL^2 - AB^2 = IN^2 - IH^2$ (776.) and $AL + AB = BM$, and $AL - AB = LB$; therefore $\overline{AL + AB} \times \overline{AL - AB} = AL^2 - AB^2 = LB \times BM = \frac{1}{4}ap$; therefore $LB = \frac{\frac{1}{4}ap}{BM}$. In the ſame Manner we prove $NH = \frac{\frac{1}{4}ap}{HO}$; wherefore, ſince HO is greater than BM, 'tis evident NH is leſs than LB; ſo that the Curve is nearer the Aſymptote at H than at B. And thus it will ever be nearer, but never meet it (by 776.)

778. Since $LB \times BM = (\frac{1}{4}ap =) NH \times HO$ (777.) It is LB : NH :: HO : BM. Draw HP, BQ ∥ CN; then are the Triangles QBM and PHO ſimilar, and therefore (657.) BM : HO :: BQ : PH (:: NH : LB.) Draw BD, RH ∥ CO; then $QB = CD$, and $PH = CR$; therefore CD : CR :: NH : LB :: RH : DB, by ſimilar Triangles NRH and DBL. Therefore $CD \times DB = CR \times RH$.

779. Draw SV ∥ CO, and TV ∥ CN; and put $SV = CS = a$, $SR = x$, and $RH = y$; then, ſince CS : CR :: RH : SV, (778.) or $a : a + x :: y : a$; it is $aa = ay + yx$, or $y = \frac{aa}{a + x}$, or (put $a = 1$) $y = \frac{1}{1 + x}$; for the Equation expreſſing the Nature of the Hyperbola between the Curve and the Aſymptotes.

780. If



780. If the Aſymptotes make a Right Angle RCF, then the Hyperbola is ſaid to be *Rectangular.* From the Point C draw CH, CB, CG, CK. The $\square$ CRHP = CDBQ (778.) Therefore the Triangle RHC = DBC; ſubtract from each the common Triangle CD*a*, there remains the Trapezium RDAH = △ C*a*B; to each of theſe add the common Space BAH, the Sums are DBHR = HBC, the Hyperbolic Sector.

781. Becauſe of Parallels, the Hyperbolic Trapezium GEFK = DBHR, and ſo the Sector CKG = CHB: and ſuppoſing the Ordinate BG to move parallelly, the Sector CVH = CVK.

782. The Line CF is divided proportionally in P Q E F, that is, CP : CQ :: CE : CF. For (778.) RH (= CP) : DB (= CQ) :: CD (= CE) : CR (= CF). Wherefore, ſince the Hyperbolic Spaces HRCP, HRCQB, HRCEG, HRCEK, are in *Arithmetical Progreſſion* (being as 1, 2, 3, 4, *&c.*) and CP, CQ, CE, CF, are in *Geometrical Progreſſion*, 'tis evident the Former are *Logarithms* of the Latter; that is, the *Hyperbolic Spaces are* Logarithms *of the Aſymptotic Abſciſſæ*, (by 130, 131.)

783. Becauſe when CP : CQ :: CE : CF, the Sector BCH = GCK; therefore when it is made CP : CQ :: CQ : CE, then the Sector BCH = BCG. Conſequently, when CP, CQ, CE, CF, are proportional, the Sector HBC = BCG = GCH. And ſo the Sectors HBC, HGA, HKC, are in *Arithmetical Progreſſion*, and therefore are *the Logarithms or Expo-*

nents of the Geometric Ratios of the Abſciſſæ. Thus, if the Ratio CP : CQ be expreſſed by the Sector HBC, then the Ratio CP : CE will be expounded by the Sector HGC, and the Ratio CP : CF, by the Sector HKC.

784. But the Sector HBC = HBDR (780.) and HBDR = HPQB. For the Rectangle RHPC = DBQC; from which ſubduct the common Rectangle D*b*PC, there remains D*b*HR = P*b*BQ; to each add the common Space H*b*B; then DBHR = HPQB. Thus the Sector BCG = QBGE; and the Sector HCG = HPEG. Whence, univerſally, the Logarithm of CE to CQ is the Space BQEG.

785. From what was demonſtrated in (769.) is deduced an eaſy mechanical Method of *deſcribing an Ellipſis, viz.* fix a Pin in each Focus, and a Third in the End of one of the Diameters, and then tying a Thread faſt about the three Pins, take out that on the End of the Diameter, and placing the Head of it within the Thread, carry it with a tight and ſteady Hand about the two Foci, and it will deſcribe the Ellipſis very exactly. But the *organical Deſcription of Curves* is a Subject that will merit more particular Attention in another Part of this Work.

786. As to the Etymology of the Names of theſe *Conic Sections*, it is derived from the Nature of the Curves reſpectively; thus when the *Square of the Semi-ordinate is equal to the Rectangle under the Parameter and Abſciſſa, viz.* $yy = px$, then that Equality is intimate by the Name of the Curve, *viz.* PARABOLA. (740.) But, when that Rectangle (px) is leſs than the Square (yy) by the Quantity $\left(\frac{pxx}{a}\right)$ then that *Deficiency* is expreſſed in the Name of the Curve ELLIPSIS. (764.) By the Word HYPERBOLA, the *Exceſs* of the Rectangle px above the Square (yy) by the Quantity $\frac{pxx}{a}$ is ſhewn. *See* (765.) It is moreover obſervable, that $\frac{pxx}{a}$ is a Rectangle, *viz.* $\frac{px}{a} \times x$ ſimilar to the conſtant Rectangle $a \times p$; for $a : p :: x : \frac{xp}{a}$. We have now premiſed the *firſt Principles of Conics*, and can enlarge upon them at Pleaſure, or as we may at any Time hereafter have Occaſion.

THE

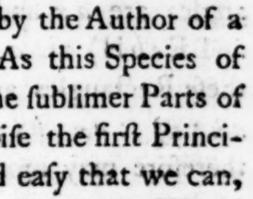
THE DOCTRINE of FLUXIONS,

OR

ELEMENTS of the NEW GEOMETRY.

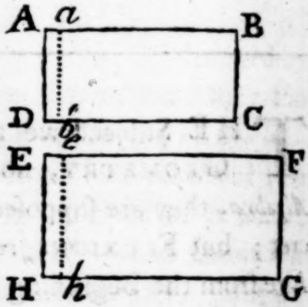
THE Subject we are now entering upon is really a NEW GEOMETRY, not in the leaſt known to the Ancients; *Algebra*, they are ſuppoſed to have underſtood; and kept it a Secret; but FLUXIONS remained among the *Incognita* of every Age from the Beginning of the World to our own. The Glory of this Invention was reſerved for the great *Britiſh* Genius, Sir ISAAC NEWTON, as is put paſt all Doubt by the Author of a Treatiſe entitled, *Commercium Epiſtolicum*. As this Species of Matheſis opens at once an Entrance to all the ſublimer Parts of Learning, it will be neceſſary here to premiſe the firſt Principles thereof in a Manner the moſt natural and eaſy that we can, and agreeable to that in which it was firſt propoſed to the World by its Author. The Doctrine of Fluxions depends upon a few Principles only, and thoſe eaſy to be underſtood, which here follow.

CHAP. XIII.



787. IF two Points A, B, be ſuppoſed to move equal Paces AM, BN, in the ſame Time, they are then ſaid to *move with the ſame or equal Velocities*; and if in every equal Portion of Time, the Spaces paſſed over, be equal, the *Velocity* is then ſaid to be *equable* or *uniform*.

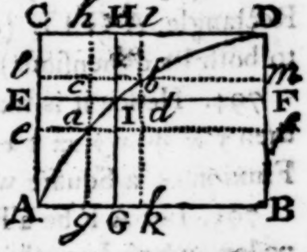
788. If in the ſame Time, the Point A paſſes from A to M, the Point B moves from B to O, then is the Velocity of the Point A to that of B, as A M to B O. Put $AM = x$, and $BO = y$. And let Aa, Bb, be the indefinitely ſmall Spaces deſcribed in the firſt Moment of Time; then the Motion or Velocities of the Points A and B being ſuppoſed equable, we ſhall have $Aa : AM :: Bb : BO$. Let $Aa = \dot{x}$, and $Bb = \dot{y}$; then ſhall $\dot{x}$, $\dot{y}$, (which are called the FLUXIONS of x and y) repreſent the Velocities with which the Points A, and B, every where move, and ſo it will be $x : y :: \dot{x} : \dot{y}$.



789. Since Superficies ABCD, EFGH, are deſcribed by the Parallel Motions of the Lines A D, E H, (615.) Let $AabD$, and $EebH$, be the indefinitely ſmall Parts deſcribed in the firſt Moment of Time in each. Put $AB = x$, $AD = a$; $EF = y$, and $EH = b$. Then alſo $Aa = \dot{x}$, and $Ee = \dot{y}$ (788.) The whole Rectangles are ax and by, and their *Naſcent Increments* ($AabD$, and $EebH$) are $a\dot{x}$ and $b\dot{y}$. If now the Times of deſcribing theſe Rectangles are equal, and the Velocities equable; then $a\dot{x} : b\dot{y} :: ax : by$; and ſo $a\dot{x}$, $b\dot{y}$, will be proportional to, and therefore may repreſent the Velocities with which thoſe Rectangles are generated, and are therefore their *Fluxions*. (788.)

790. But what we have now ſaid relates to the Fluxion or Flowing of the Rectangle *one Way only, or according to one Dimenſion*; but ſuppoſing it to increaſe according to both its Dimenſions, or ſo that both its Sides may flow to compleat the Space it contains, then we muſt add the Fluxions of both Sides together, and their Sum will be the Fluxion of the Rectangle in this Caſe.

791. In order to illuſtrate this it muſt be conſidered, that the illuſtrious Author of this Invention firſt delivered the Idea of what we now call a *Fluxion* under the Name of *Momentum*, which was a Term uſed in *Mechanics* to denote the Quantity of Motion generated by a given Quantity of Matter (A), and the Velocity *(a)* with which it moved conjointly. This *Momentum* therefore was properly repreſented by (A*a*); and if (B*b*) denote any other *Momentum* reſulting from a Quantity of Matter



(B) moving with any Velocity as (*b*) hen if theſe *Momenta* are generated in he ſame time, they will be as A *a* to B*b*; and if the Velocities of Motion re equal in both A and B, then will he *Momenta* be as the Quantities A nd B. But inſtead of this mechanial Notation, we now uſe $x\dot{x}$ and $y\dot{y}$ for the *Momenta*, or Fluxons, generated by the Quantities *x* and *y* as above deſcribed.

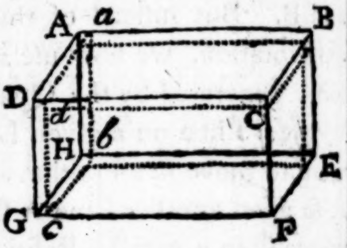
792. Then on a given Line AB, ſuppoſe a right Line GH ere to move in a Poſition always parallel to itſelf, from A to B. lſo upon another Line AC let the Line EF move from A towards C in a parallel Poſition alſo. And let the Velocities of he moving Lines be ſuch, that their Interſection at I ſhall deſcribe the Curve Line AID. By the Motion of theſe Lines here will be deſcribed the Curvilineal Spaces AIDB and IDC, alſo the Parallellograms AEFB and ACHG. Put G = (EI =) *x*, and GI = (AE =) *y*; and the Fluxions f the Spaces deſcribed by the Generating-Lines *x* and *y*, will e always as the Magnitude of thoſe Lines, ſuppoſing the Velocity of each to be conſtant. Therefore when the Line AC arrives o the Poſitions *gh*, GH, *ik*, the Fluxion of the curve-lined Space AIDB will be as *ag*, IG, and *bk*; but the Fluxion f the Parallellogram AEFB will be in each Poſition the ſame, eing always as the equal Lines *cg*, IG, and *dk*. In the irſt Poſition therefore at *gh*, the Fluxion of the Space AIDB s leſs than that of the Parallellogram, becauſe *ag* is leſs than *g*. In the third Poſition at *ik*, the Fluxion *bk* of the ſaid Space is greater than *dk* the Fluxion of the Parallelogram, onſequently there muſt be ſome intermediate Poſition GH where the Fluxion GI is the ſame in both.

793. In like manner it is ſhewn, that the Line which moves rom AB to CD, when it is in the Poſitions *ef*, EF, and *lm* produces the Fluxions of the Space AIDC leſs, equal to, and greater than thoſe of the Parallellogram AGHC. But we ave ſhewn that the Fluxion of the Rectangle ABFE is at he Poſition at IG equal to $y\dot{x}$, and of the Rectangle AGHC t is $x\dot{y}$; but theſe are alſo equal to the Fluxions of the two urvilineal Spaces AIG and AIE, which together make the

Rect-

Rectangle AGIE; therefore the whole Fluxion of the ſaid Rectangle AGIE (or xy) (conſidered as flowing according to both its dimenſions) is equal $y\dot{x} + x\dot{y}$.

794. Hence it is manifeſt that when AG = AE, or $x = y$, then $y\dot{x} + x\dot{y} = 2x\dot{x}$, or $2y\dot{y}$; which therefore is as th Fluxion of a Square whoſe Side is x or y, that is of x^2 or y^2.



795. Let AF be a Parallelopipedon, whoſe Length is $AB = x$, its Breadth $AD = y$, and Depth $AH = z$. Suppoſe it to flow equably according to its Length only, by the Motion of the Plane $ADGH = yz$, and ſuppoſe $Aa\,(= \dot{x})$ the Space deſcribed in the Direction of AB th firſt Moment of Time; then will the Solid A*c* be as the Flux-ion of the Solid AF, and therefore equal to $yz\dot{x}$. If the So-lid were to flow only according to its Dimenſion of Breadth AB, by the Motion of the Plane $ABEH = xz$, then would the Fluxion be $xz\dot{y}$. And if the Solid flows in the Dimen-ſion of Depth AH, by the Plane $ABCD = xy$, then is it Fluxion $xy\dot{z}$. If then it flows in all its three Dimenſions, the momentary Increaſe or Decreaſe will be proportional to th Sum of all, *viz.* $yz\dot{x} + xz\dot{y} + xy\dot{z}$, which therefore is th Fluxion of the whole Solid in this Caſe.

796. The ſame Thing may be otherwiſe proved thus: Pu $xy = v$; then $xyz = vz$; and by (791, 792, 793,) $\dot{v} = \dot{x}y +$ $x\dot{y}$, and the Fluxion of vz, *viz.* $\dot{v}z + v\dot{z}$ is = the Fluxio of xyz: Therefore, by Reſtitution, or writing xy for v, an $\dot{x}y + x\dot{y}$ for $\dot{v}$, we ſhall have the Fluxion of $xyz = \dot{x}yx +$ $x\dot{y}z + xy\dot{z}$ as before. And when $x = y = z$, then $xyz = x$ and its Fluxion $3x^2\dot{x}$, or $3xx\dot{x}$.

797. Since the Fluxion of x is $\dot{x}$ (788); and of x^2 is $2x\dot{x} =$ $2x^{2-1}\dot{x}$ (794); and of x^3, is $3x^2\dot{x}$ (796) $= 3x^{3-1}\dot{x}$; it i evident the Fluxion of x^m is $mx^{m-1}\dot{x}$; and the Fluxion of $x^m y$ is $mx^{m-1}\dot{x}y^n + ny^{n-1}\dot{y}x^m$.

798. Becauſe $\sqrt{x} = x^{\frac{1}{2}}$ (for $x : \sqrt{x} :: \sqrt{x} : 1$, and x^1 $x^{\frac{2}{2}} :: x^{\frac{1}{2}} : x^0$;) therefore the Fluxion of $\sqrt{x} = x^{\frac{1}{2}}$ is $\frac{1}{2}x^{\frac{1}{2}-1}$ $= \frac{1}{2}x^{\frac{1-2}{2}}\dot{x} = \frac{1}{2}x^{-\frac{1}{2}}\dot{x}$; and in like Manner the Fluxio

of $\sqrt[2]{x^3} = x^{\frac{3}{2}}$, is $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$; and in general the Fluxion of $\sqrt{x^m} = x^{\frac{m}{n}}$ is $\frac{m}{n}x^{\frac{m-n}{n}}\dot{x}$.

799. Thus alſo the Fluxion of $\sqrt[2]{xy} = x^{\frac{1}{2}}y^{\frac{1}{2}}$, is $\frac{1}{2}x^{-\frac{1}{2}}$ $\dot{x}y^{\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\dot{y}x^{\frac{1}{2}}$, or $\frac{\dot{x}y^{\frac{1}{2}}}{2x^{\frac{1}{2}}} + \frac{\dot{y}x^{\frac{1}{2}}}{2y^{\frac{1}{2}}}$; and in general, the Fluxion of $\sqrt[m]{x^n y^p} = x^{\frac{n}{m}}y^{\frac{p}{m}}$, is $\frac{p}{m}x^{\frac{n-m}{m}}\dot{x}y^{\frac{p}{m}} + \frac{p}{m}y^{\frac{p-m}{m}}$ $\dot{y}x^{\frac{n}{m}}$.

800. The Fluxion of any Root of *Surd Compound* Quantities are alſo thus eaſily obtained. As the Fluxion of $\sqrt[2]{xy+yy}$ $= \overline{xy+yy}^{\frac{1}{2}}$ is $\frac{1}{2}\overline{xy+yy}^{-\frac{1}{2}} \times \overline{x\dot{y}+y\dot{x}+2y\dot{y}} =$ $\frac{x\dot{y}+y\dot{x}+2y\dot{y}}{2\overline{xy+yy}^{\frac{1}{2}}}$. And univerſally the Fluxion of $\sqrt[q]{a^n+x^m}$ $= \overline{a^n+x^m}^{\frac{1}{q}}$ is $\frac{1}{q}\overline{a^n+x^m}^{\frac{1-q}{q}} \times mx^{m-1}\dot{x}$.

Alſo $x^p\sqrt[q]{y^n+z^m} = x^p \times \overline{y^n+z^m}|^{\frac{1}{q}}$ gives the Fluxion

$$x^{p-1}\dot{x} \times \overline{y^n+z^m}|^{\frac{1}{q}} + x^p \times \frac{\overline{ny^{n-1}\dot{y}+mz^{m-1}\dot{z}}}{q \times \overline{y^n+z^m}|^{\frac{q-1}{q}}}$$

801. The Fluxion of a Fraction $\frac{x}{y}$ is thus found. Put $= v$, then $x = vy$, and the Equality between x and vy ught ever to hold in all their State of Increaſe or Decreaſe, heir Fluxions will always be equal, but the Fluxion of x is $\dot{x}$, nd the Fluxion of vy is $v\dot{y}+y\dot{v}$ (792, 793); therefore $\dot{x}=v\dot{y}$ $+y\dot{v}$; and ſo $\dot{x}-v\dot{y}=\dot{y}\dot{v}$, and $\dot{v} = \frac{\dot{x}-v\dot{y}}{y} = \frac{y\dot{x}-x\dot{y}}{yy}$ y ſubſtituting for v its Value $\frac{x}{y}$.

802. Becauſe (a) repreſents a *conſtant Quantity*, its Fluxion is (o), therefore the Fluxion of $a + x$ is only $\dot{x}$, and the ſame may be ſaid of numerical Quantities, thus the Fluxion of $1 + x$ is only $\dot{x}$. The Fluxion of $a x$, or $2 x$, is only $a \dot{x}$ or $2 \dot{x}$. Hence the Fluxion of $\frac{a}{x}$ is $\frac{-a\dot{x}}{xx}$, and that of $\frac{x}{a+x}$ is $\frac{a\dot{x}}{aa + 2ax + xx}$. But the Fluxions o complicated Fractions are beſt found in their Value expreſſed i infinite Series. Thus $\frac{aa}{b+x} = \frac{a^2}{b} - \frac{a^2 x}{b^2} + \frac{a^2 x^2}{b^3}$, &c. (103); th Fluxion of which Series is $-\frac{a^2\dot{x}}{b^2} + \frac{2a^2 x\dot{x}}{b^3} - \frac{3a^2 x^2\dot{x}}{b^4}$, &c (by 801).

803. But the principal Uſe of Fluxions is to find the *Fluen* or *flowing Quantity* from the *Fluxion given*, which is done in general by the Reverſe of the Method of finding the Fluxion o a given Fluent, above delivered. This Method conſiſts in thre Particulars, *viz.* (1) Expunge the fluxionary Letter; thus th Fluxion $2 x \dot{x}$ becomes $2 x$. (2) Add Unity to the Index o Exponent of the flowing Quantity; and thus $2 x$ become $2 x^{1+1} = 2 x^2$. (3) Divide by the Exponent thus encreaſe and ſo $2 x^2$ becomes x^2, which is the Fluent of the Fluxio $2 x \dot{x}$, as required. Thus $3 x^2 \dot{x}$ is (1) $3 x^2$, (2) $3 x^3$, and (3 x^3. So in general $m x^{m-1} \dot{x}$ becomes firſt $m x^{m-1}$, then $m x^m$ and laſtly x^m the Fluent required.

804. Hence alſo the Fluent of $x \dot{x}$ is $\frac{xx}{2}$; for it is firſt x then x^2; and laſtly $\frac{x^2}{2}$. So $x^2 \dot{x}$ is firſt, x^2; ſecondly x^3; thirdly $\frac{x^3}{3}$ the flowing Quantity; and by Inſpection it ap pears that the Fluent of $\dot{x}$ is x; of $x\dot{y} + y\dot{x}$ is $x y$ (by 79 793,) and the Fluent of $x y \dot{z} + x z \dot{y} + \dot{x} y z$ is $x z y$ (b 796.) And alſo that the Fluent of $\frac{x\dot{y} - y\dot{x}}{yy}$ is $\frac{x}{y}$; and that th Fluent of $\dot{x} + x\dot{x} + x^2\dot{x}$, &c. is $x + x^2 + x^3$, and in ge

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neral the Fluent of $mx^{m-1}\dot{x}$ will be x^m; and of $-mx^{-m-1}\dot{x}$ will be x^{-m}, or $\frac{1}{x^m}$ (798).

805. Thus alſo the Fluent of $\frac{m}{n}x\dot{x}^{\frac{m-n}{n}}$ is $x^{\frac{m}{n}}=\sqrt[n]{x^m}$, and of $ax^{\frac{m}{n}}\dot{x}$ is $\frac{an}{m+n}x^{\frac{m+n}{n}}$ for by rejecting $\dot{x}$, it is $ax^{\frac{m}{n}}$, and by adding 1 to the Index $\frac{m}{n}$ it is $ax^{\frac{m+n}{n}}$ (for $\frac{m}{n}+1=\frac{m+n}{n}$;) and dividing by $\frac{m+n}{n}$, we have $\frac{an}{m+n}x^{\frac{m+n}{n}}$ as above. Now this is a general Form for any other fluxionary Power or Root, as $4\sqrt{x\dot{x}}=4x^{\frac{1}{2}}\dot{x}$, where $a=4$, $m=1$, $n=2$, and ſo $\frac{m+n}{n}=\frac{3}{2}$, therefore $\frac{an}{m+n}=\frac{8}{3}$; and conſequently we ſhall have $\frac{an}{m+n}x^{\frac{m+n}{n}}=\frac{8}{3}x^{\frac{3}{2}}=\frac{8}{3}\sqrt[2]{x^3}$ for the Fluent of $4x^{\frac{1}{2}}\dot{x}$. After this Manner the Fluent of $\sqrt[3]{x^5\dot{x}}=x^{\frac{5}{3}}\dot{x}$ will be found $\frac{3}{8}x^{\frac{8}{3}}=\frac{3}{8}\sqrt[3]{x^8}$; and of $x^{\frac{1}{2}}\dot{x}=1x^{-2}\dot{x}$, will be $-x^{-1}$; of $\frac{1}{\sqrt{x^3}}\dot{x}=x^{-\frac{3}{2}}\dot{x}$, will be $-\frac{2}{1}x^{-\frac{1}{2}}=\frac{2}{-\sqrt{x}}=$. And laſtly, the Fluent of $\frac{1}{x}\dot{x}=x^{-1}\dot{x}$ will be $\frac{1}{0}x^{\frac{0}{1}}=\frac{1}{0}x^0=\frac{1}{0}\times 1=\frac{1}{0}=$ Infinity; ſince here $a=1$, $m=-1$, and $n=1$.

806. If a Fluxion be multiplied into ſome Quantity affected with a *Vinculum*, that is, into ſome Compound or Surd Quantity, which Quantity is the Fluent of the ſaid Fluxion, as $a\dot{x}\times\overline{aa+ax}|^m$, then will the Fluent of the Expreſſion be had by the general Rule thus; ſtrike out the Fluxion, and it becomes $\overline{aa+ax}|^m$; and add Unity to the Index m, and it is $\overline{aa+ax}|^{m+1}$; then divide by the Index ſo increaſed, and you

have the Fluent $\frac{1}{m+1} \times \overline{aa+ax}|^{m+1}$ for the Expreſſion propoſed.

807. Thus alſo $a\dot{x} \times \sqrt[n]{aa+ax}|^{m} = a\dot{x} \times \overline{aa+ax}|^{\frac{m}{n}}$, by the very ſame Proceedure, will have its Fluent thus expreſſed $\frac{n}{n+m}\,\overline{aa+ax}|^{\frac{m+n}{n}}$. Thus the Fluent of $2x\dot{x} \times \sqrt{aa+xx} = 2x\dot{x} \times \overline{aa+xx}|^{\frac{1}{2}}$ is $\frac{2}{3}\,\overline{aa+xx}|^{\frac{3}{2}}$, or in Surds $\frac{2aa+2xx}{3}\sqrt{aa+x^2}$. So the Fluent of this Expreſſion $mx^{m-1}\dot{x} \times \overline{a^q+x^m}|^{n}$ is $\frac{n+1}{1}\,\overline{a^q+x^m}|^{n+1}$; and ſo for any other of this kind.

808. If the Fluxion without the *Vinculum* be not that of the Quantity under it, but is yet *in ſome given Ratio* to it, the Fluent may in this Caſe be had in a finite Number of Terms. Suppoſe the Expreſſion were $x\dot{x}\sqrt{aa+x^2}$, or $x\dot{x} \times \overline{aa+xx}|^{\frac{1}{2}}$ where the Fluxion $x\dot{x}$ is to that of x^2, as 1 to 2, *viz.* $x\dot{x} : 2x\dot{x} :: 1 : 2$. It will be beſt here to proceed by Subſtitution; therefore let $\sqrt{aa+x^2} = z$; then $aa + xx = zz$, and $2x\dot{x} = 2z\dot{z}$, or $x\dot{x} = z\dot{z}$; therefore $x\dot{x}\sqrt{aa+xx} = zz\dot{z}$; but the Fluent of $zz\dot{z}$ is $\frac{1}{3}z^3 = \frac{1}{3}zz \times z =$ (by Subſtitution) $\frac{aa+xx}{3}\sqrt{aa+xx}$.

809. Alſo the Fluent of $x^{m-1}\dot{x} \times \overline{aa^q+x^m}|^{n}$ may be found in like Manner. For here the Fluxion $x^{m-1}-\dot{x}$ is to that under the *Vinculum* of x^m, *viz.* $mx^{m-1}\dot{x}$, as 1 to m; which Ratio is given. Therefore put $\overline{aa^q-x^m}|^{n} = z^n$, and ſo $aa^q+x^m = z$, and conſequently $mx^{m-1}\dot{x} = \dot{z}$, and therefore $x^{m-1}\dot{x} = \frac{\dot{z}}{m}$, therefore $x^{m-1}\dot{x}\,\overline{\times aa^q+x^m}|^{n} = \frac{z^n\dot{z}}{m}$. But the Fluent of the Latter, *viz.* $\frac{z^n\dot{z}}{m} = \frac{1}{m} \times z^n\dot{z}$ is $\frac{1}{mn+n}z^{n+1} =$

$\frac{1}{mn+n}\overline{\times\, a a^{q} x^{m}}\big|^{n+1}$ = the Fluent of the given Expreſſion.

810. Thus alſo the Fluent of $d x^{m-1}\dot{x} \times \overline{a+fx^{m}}\big|^{n}$ may be found; for put $\overline{a+fx^{m}}\big|^{n} = z^{n}$, then $\overline{a+fx^{m}} = z$, and ſo $\dot{z} = fm x^{m-1}\dot{x}$, and therefore $\frac{\dot{z}}{fm} = x^{m-1}\dot{x}$, and ſo $\frac{d\dot{z}}{fm} = dx^{m-1}\dot{x}$; therefore $dx^{m-1}\dot{x}\times\overline{a+fx^{m}}\big|^{n} = \frac{d\dot{z}}{fm}\times z^{n} = \frac{d}{fm}\times z^{n}\dot{z}$, but the Fluent of this latter Fluxion is $\frac{d}{fm}\times\frac{1}{n+1}\times z^{n+1}$, or $\frac{d}{fm\times\overline{n+1}}\, z^{n+1}$; if therefore inſtead of z we ſubſtitute its Equal $\overline{a+fx^{m}}$, we have $\frac{d}{fm\times\overline{n+1}}\overline{\times\, a+fx^{m}}\big|^{n+1}$ for the Fluent of the Expreſſion propoſed.

811. The laſt is a *general Form* for any fluxional Expreſſion of that Kind; thus ſuppoſe the Fluent of $\dot{x}\times\overline{a+x}$ were required; here $m=1$, $n=1$, $f=1$, $dx^{m-1} = dx^{1} = d = 1$, therefore $\frac{d}{fm\times\overline{n+1}} = \frac{1}{2}$, and conſequently $\frac{1}{2}\overline{a+x}\big|^{2}$, will be the Fluent required. Again, let $x\dot{x}\sqrt{aa+x^{2}} = x\dot{x}\overline{\times\, aa-x^{2}}\big|^{\frac{1}{2}}$; here $m=2$, $n=\frac{1}{2}$, $f=1$, and $dx^{m-1} = x = dx$, and therefore $d=1$; hence $\frac{d}{fm\times\overline{n+1}} = \frac{1}{2\times\frac{3}{2}} = \frac{1}{3}$, and ſo $\frac{1}{3}\overline{aa+x^{2}}\big|^{\frac{3}{2}}$ or $\frac{aa+x^{2}}{3}\sqrt{aa+x^{2}}$ is the Fluent, the ſame as before.

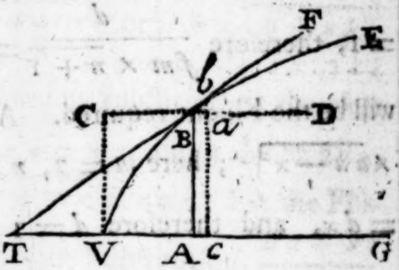
812. The Fluents of ſuch fluxional Expreſſions may be found alſo by an infinite Series; for $\overline{a+fx^{m}}\big|^{n}$, thrown into an infinite Series will be $a^{n} + \frac{n}{1}fx^{m}a^{n-1} + \frac{n}{1}\times\frac{n-1}{2}\times ff x^{2m}a^{n-2}\times$, &c. Or if $A = \frac{1}{a}$, $B = A\frac{n}{a}$, $C = B\frac{n-1}{2a}$, $D = C\frac{n-2}{3a}$, &c. we ſhall have $\overline{a+fx^{m}}\big|^{n} = a^{n}\times 1 + n$

$\mathrm{A} f x^m + \frac{n-1}{2} \mathrm{B} f f x^{2m} + \frac{n-2}{3.} \mathrm{C} f^3 x^{3m} +$, &c. Now if this Series be multiplied by the Fluxion $d x^{r-1} \dot{x}$, and the Fluent of each Term be taken, we ſhall have $a^n d x^r \times \frac{1}{r}$

$$+ \frac{n}{r+m} \mathrm{A} f x^m + \frac{n-1}{r+2m} \tfrac{1}{2} \mathrm{B} f^2 x^{2m} + \frac{n-2}{r+3^m} \tfrac{1}{3} \mathrm{C} f$$

x^{3m}, &c. for the Fluent of $d x^{r-1} \dot{x} \times \overline{a + f x^m}|^n$ which therefore will ſerve as a general Form for all binomial Expreſſions of Fluxions of the ſame Kind with this. But where the Fluent can be had in finite Terms, the general Form of the (810) is much the beſt.

813. But it muſt be confeſſed there is no direct and certain Method of finding the Fluent of a given Fluxion that ſhall hold in every Caſe, becauſe there may be different Fluents which produce, and conſequently which belong to the ſame Fluxion; thus x, $a + x$, $x + 1$, &c. all have the common Fluxion $\dot{x}$; and if there be nothing in the Circumſtances or Conditions of the Problem to determine the Fluent, we are wholly uncertain what it is. But of this more hereafter.



814. To find the Fluxion of *Curve Lines* and *Curvilineal Spaces*, or Areas, there is required but one *Poſtulatum*, viz. *that the Curve* V B E *and its Tangent* T F *at the Point of Contact* B, *are in the ſame Direction for an indefinite ſmall Diſtance on either Side, as* B b. *Or, that the ſmall Portion of the Curve and Tangent* B b *are coincident, and make a Right Lineola*; then putting the Abſciſs V A $= x$, the Ordinate A B $= y$, and the Curve V B $= z$, and draw $c b$ infinitely near and parallel to A B, and C B a parallel to the Axis T G. If now we ſuppoſe the Ordinate A B to move equably along the Axis, and with its Point B to deſcribe the Curve V B E, then it is evident in any Situation of the Ordinate A B, the Fluxion, or Velocity, of the Abſciſs A V, is A $c = \dot{x}$; and of the Ordinate A B, it is $a b = \dot{y}$; and of the Curve V B, it is B $b = \dot{z}$.

815. Now becauſe of parallel Lines, the Triangles Bab and TAB are ſimilar; and therefore (657) $TA : AB :: Ba : ab :: \dot{x} : \dot{y}$. Let the Curve be the *Parabola*; then $TA = 2x$, (745) and ſo $2x : y :: \dot{x} : \dot{y}$ or $2x\dot{y} = y\dot{x}$; but $px = yy$, (740) and putting $p = 1$, it is $x = yy$, and ſo $2x\dot{y} = 2yy\dot{y} = y\dot{x}$, or $2y\dot{y} = \dot{x}$. Whence it appears again, that as the Fluxion of x is $\dot{x}$, ſo the Fluxion of yy is $2y\dot{y}$ agreeably to (794).

816. Becauſe the Ordinate AB is at Right Angles with the Axis, the ſmall Triangle Bab is rectangular, and ſo $Ba^2 + ab^2 = Bb^2$, $\dot{x}^2 + \dot{y}^2 = \dot{z}\dot{z}$, conſequently the Fluxion of any Curve VB is $\dot{z} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$.

817. Again it is evident, the Fluxion of the curvilineal Space AVB is the ſame with that of the Parallellogram $AVCB$, at the Inſtant the Ordinate arrives at the Situation AB; for at any Time before it was leſs, and at any Time after, it is greater. But the Fluxion of the Parallellogram is as $AcaB = y\dot{x}$, (792) which therefore is the Expreſſion for the Fluxion of any curvilineal Space.

CHAP. XIV.

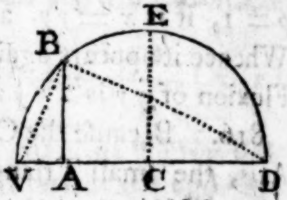
Of the METHOD *de* MAXIMIS *and* MINIMIS; *the* RECTIFICATION *of* CURVES; *the* QUADRATURE *of* Curvilineal Spaces; *and the* CUBATURE *of* SOLIDS.

818. WE ſhall now give a Specimen of the univerſal Application of Fluxions to all mathematical Purpoſes; and which will contain the original Principles of the moſt extenſive Geometry: We ſhall illuſtrate this Subject in the following capital Articles.

I. *The* METHOD *de* MAXIMIS *and* MINIMIS.

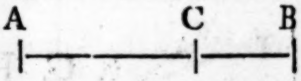
This Method conſiſts in determining an *extreme Value* of any propoſed Quantity, that is, to find when it is *greateſt*, or *leaſt*.

Now, as ſuch a Quantity becomes greater or leſſer *by flowing*, ſo when it has attained its greateſt or leaſt Magnitude, that flowing State is at an End, and its Fluxion in that Caſe muſt of Courſe be nothing, or $= 0$.

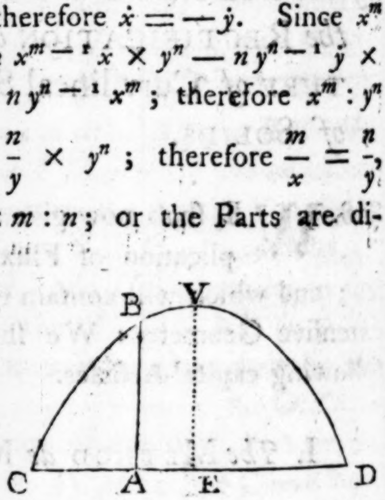


819. To illuſtrate this, let V B D be a Circle; $VD = a =$ Diameter, the Abſciſs $VA = x$, the Ordinate $AB = y$; then $AD = a - x$, and becauſe of the Rect-angle VBD, (645) it is $AD : AB :: AB : AV$, that is $a - x : y :: y : x$, and ſo $ax - xx = yy$. Now to find when A B or y is greateſt, put the Equation into Fluxions, and it is $a\dot{x} - 2x\dot{x} = 2y\dot{y}$. And becauſe in this Caſe $\dot{y} = 0$, (818) therefore $2y\dot{y} = 0$, and ſo $a\dot{x} - 2x\dot{x} = 0$. Hence $a\dot{x} = 2x\dot{x}$, that is, $a = 2x$, or $x = \frac{1}{2}a = VC$, when y or A B is a *Maximum*, as it evidently is at CE.

820. Hence it appears, that when a Line V D is divided into two Parts, A V and A D, whoſe Product $AV \times AD$ is a *Maximum*, thoſe Parts are *equal to each other*, or the given Line is *biſected*.



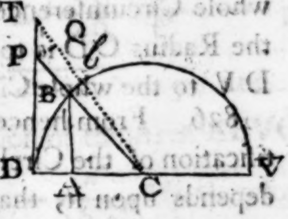
821. Let the right Line A B be ſo divided in C, that $AC^m \times CB^n$, may be a *Maximum*; put $AC = x$, and $CB = y$; then it is evident, that the Fluxion of both Parts is the ſame, and that while A C increaſes, B C decreaſes; therefore $\dot{x} = -\dot{y}$. Since $x^m y^n$ is a *Maximum*, its Fluxion $mx^{m-1}\dot{x} \times y^n - ny^{n-1}\dot{y} \times x^m = 0$; whence $mx^{m-1}y^n = ny^{n-1}x^m$; therefore $x^m : y^n :: mx^{m-1} : ny^{n-1} :: \frac{m}{x} \times x^m : \frac{n}{y} \times y^n$; therefore $\frac{m}{x} = \frac{n}{y}$, and $nx = my$, whence $x : y :: m : n$; or the Parts are directly as their Powers.



822. Again, let C V D be a Parabola; and put $CD = a$, $CA = x$, $AB = y$, then $AD = a - x$, and (762) $p \times AB = CA \times AD$, or $py = ax - xx$; and in Fluxions $p\dot{y} = a\dot{x} - 2x\dot{x}$. When therefore A B or y is a *Maximum*, it is $\dot{y} = 0$; then $a\dot{x} = 2x\dot{x}$, or $x = \frac{1}{2}a = CE$, as before.

II. *The* RECTIFICATION *of* CURVES.

823. By the *Rectification* of a Curve, is meant no more than the finding a *right Line equal to it.* I ſhall give an Example in the Arch D B of the Circle V B D. Let T D be a Tangent at D, draw C P, and infinitely near it C T; and with the Radius C P deſcribe the ſmall Arch P Q. Put D C $= a$, D P $= x$, and then T P $= \dot{x}$, and B $b = \dot{z}$ = Fluxion of the Arch D B. Then it is T P : Q P :: C P : C D; and P Q : B b :: C P : C B = C D. Therefore (*ex Æquo* 652,) T P : B b :: C P² : C D²; but C P² $= a^2 + x^2$; therefore $a^2 + x^2 : a^2 :: \dot{x} : \dot{z}$; whence $\dot{z} = \frac{a^2 \dot{x}}{a^2 + x^2}$; now $\frac{a^2}{a^2 + x^2} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6}$, &c. which multiplied by $\dot{x}$, makes $\dot{x} - \frac{x^2 \dot{x}}{a^2} + \frac{x^4 \dot{x}}{a^4} - \frac{x^6 \dot{x}}{a^6}$, &c. $= \frac{a a \dot{x}}{a a + x^2} = \dot{z}$, the Fluent of which is $x - \frac{x^3}{3 a^2} + \frac{x^5}{5 a^4} - \frac{x^7}{7 a^6}$, &c. = D B the Arch required in the Values of the right Lines C D, and D P.



824. By putting C D $= a = 1$, the Length of the Arch D B will be thus expreſſed, $x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7$, &c. And if the Arch B D = 30 Degrees, then A B $= \frac{1}{2}$ C D $= \frac{1}{2}$, * and A C² = C B² − A B² $= 1 - \frac{1}{4} = \frac{3}{4}$ whence A C $= \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$. And ſince C A : A B :: C D : D P, or $\frac{1}{2}\sqrt{3} : \frac{1}{2} :: 1 : x = \frac{1}{\sqrt{3}}$, and ſo $x^2 = \frac{1}{3}$, and $x^3 = \frac{1}{3}\sqrt{\frac{1}{3}}$, $x^5 = \frac{1}{9}\sqrt{\frac{1}{3}}$, $x^7 = \frac{1}{27}\sqrt{\frac{1}{3}}$, &c. Therefore by ſubſtituting theſe Values of x, we have the Arch D B $= \sqrt{\frac{1}{3}} - \frac{1}{9}\sqrt{\frac{1}{3}} + \frac{1}{45}\sqrt{\frac{1}{3}} - \frac{1}{189}\sqrt{\frac{1}{3}}$, or D B $= \sqrt{\frac{1}{3}} \times 1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729}$, &c.

825. Now ſince D B is $\frac{1}{6}$ of the Semi-circle; therefore $6\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{12}} = 2\sqrt{3} = 3{,}464101615$; and hence, if each

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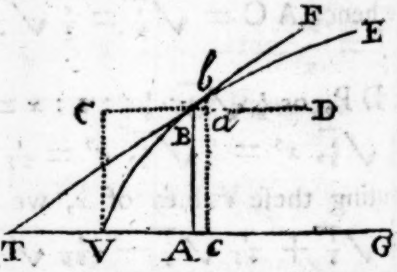
* For the Side of an Hexagon is the Chord of 60 Degrees, and equal to the Radius; Half that Chord is the Sine of 30 Degrees, which is therefore equal to Half the Radius.

Term in the above Series be multiplied by this Number, and the Sum of all the negative Products taken from the Sum of all the affirmative Ones, the Reſidue will give the Value of the whole Circumference $= 3,14159265358979$, &c. Wherefore the Radius C D is to the Semi-circle D B V (or the Diameter D V to the whole Circle) as 1 to 3,14159, &c.

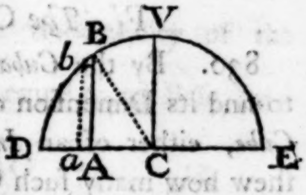
826. From hence appears the Impoſſibility of an exact Rectification of the Circle; and ſince the Quadrature of the Circle depends upon it, that alſo is impoſſible; in vain therefore muſt any one attempt to *rectify, or ſquare the Circle*, tho' we can approximate to the Truth, as there can ever be Occaſion for. The following prodigious Number, calculated by the late Mr. *Sharp, viz.* 3,1415 9265 3589 7932 3846 26433832 7950 2884 1971 6939 9375 1058 2097 4944 5923 0781 6405; and afterwards examined to the laſt Figure by Mr. *Machin*, is more than ſufficient for computing the Number of Grains of Sand that may be contained within the Sphere of the fixed Stars. I give this Number a Place here as it is the Product of the greateſt Effort that was ever made in practical Geometry. The Rectification of the *Conic Sections*, and other Curves, will hereafter find a Place, when we have Occaſion to apply them to Uſe.

III. *The* QUADRATURE *of* CURVILINEAL SPACES.

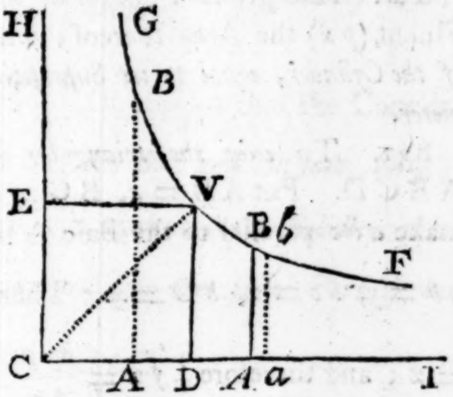
827. To ſquare any Curve-lined Space, is to find an Expreſſion of a Right-lined Space equal to it. Thus, ſuppoſe it is required to ſquare the *Parabolic Area* ABV. Then becauſe $px = yy$, (658) we have $y = \sqrt{px} = p^{\frac{1}{2}} x^{\frac{1}{2}}$, which multiplied by $\dot{x} =$ Ac, gives $y\dot{x} = p^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} =$ A$cb$B, the Fluxion of the Space. (789) The Fluent of which is $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$ (803) $= \frac{2}{3} \sqrt{px^3} = \frac{2}{3} \sqrt{px \times x^2} = \frac{2}{3} = \sqrt{y^2 \times x^2} = \frac{2}{3} yx$. But CB $=$ VA $= x$, CV $=$ AB $= y$; *wherefore the Area* A B V *of the Parabola is* $\frac{2}{3}$ *of its circumſcribing Parallellogram* A B C V. And conſequently the external Space VBC is $\frac{1}{3}$ thereof.



828. To ſquare the Circle: Let the Ordinate A B move from V C equably toward D put C V = B C = a, A C = x, A B = y, then A C^2 + A B^2 = B C^2, or $a a = x^2 + y^2$. Therefore $a a - x^2 = y^2$, and $\sqrt{a^2 - x^2} = y$, and $\dot{x} \sqrt{a a - x^2} = y \dot{x} =$ A a b B, the Fluxion of the Area ABVC, (789). But $\sqrt{aa - xx} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$, *&c.* therefore $\dot{x} \sqrt{a d - x^2} = a \dot{x} - \frac{x^2 \dot{x}}{2a} - \frac{x^4 \dot{x}}{8a^3} - \frac{x^6 x}{16a^5}$, *&c.* $= y \dot{x}$; the Fluent of which Series is $a x - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5}$, *&c.* = A B V C, the Space required in Terms of the right Line A C. When A B arrives at D and vaniſhes, then $a = x =$ C D = 1, and the Series will become $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} -$, *&c.* = the Quadrant V C D; or if the Diameter = 1 = D E, then this Series will be the Area of the whole Circle, and will be = 0,785398, *&c.*



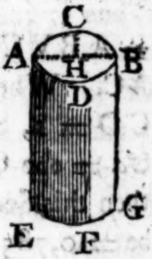
829. To ſquare the *Hyperbolic Space* DVBA; let CD = a, DA = x, AB = y; then it is $y = \frac{aa}{a + x}$, (or putting $a = 1$,) $y = \frac{1}{1 + x}$. (239) The Fluxion of which is $\dot{x}\, y = \frac{\dot{x}}{1 + x} =$ A a b B the Fluxion of the Space A D V B. But $\frac{1}{1 + x} = 1 - x + x^2 - x^3$, *&c.* (as will appear by Diviſion) therefore $\frac{\dot{x}}{1 + x} = \dot{x} - x \dot{x} + x^2 \dot{x} - x^3 \dot{x}$, *&c.* the Fluent of which is $x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4$, *&c.* = A D V B, the Space required.



IV. *The* CUBATURE *of* SOLIDS.

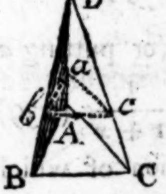
830. By the *Cubature of a Solid* is meant no more than to find its Dimenſion or Content eſtimated in the Meaſure of a *Cube*, either of an *Inch*, *Foot*, *Yard*, *Mile*, &c. that is, to ſhew how many ſuch Cubes are contained in, or are equal to the Bulk of the given Solid; to which Purpoſe the following Theorems are premiſed.

In the Circle DBV (Fig. to Art. 823.) the infinitely ſmall Sector CBb may be eſteemed a *rectilineal Triangle*, and ſo equal to $\frac{1}{2}BC \times bB$, or $\frac{1}{2}a\dot{z}$, the Fluent of which is $\frac{1}{2}az =$ the Sector CDB; and putting $p =$ Periphery of the Circle, when $z = p$, then the Area of the Circle is $\frac{1}{2}ap$. This being premiſed, we proceed



831. To *cube the* CYLINDER AG. Put $AH = a$, and the Circumference $= p$; then will the circular Area be $\frac{1}{2}ap$, and let $AE = x$; then $\frac{1}{2}ap\dot{x}$ will be the Fluxion of the Cylinder, and the Fluent thereof $\frac{1}{2}apx =$ the Solidity of the Cylinder.

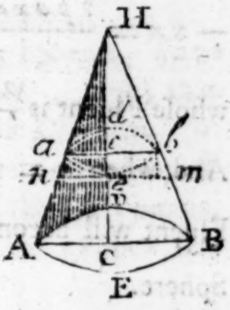
832. Alſo $p\dot{x} =$ Fluxion of the cylindric Surface, and its Fluent (px) the Area thereof; whence $\frac{1}{2}a \times px =$ the *Solidity of the Cylinder, equal to its Superficies multiplied by half its Diameter.*



833. To *cube the triangular* PYRAMID ABCD. Put $AB = a$, $BC = b$, $BD = d$; make abc parallel to the Baſe ABC; and put $ab = y$, $bc = z$, $bD = x$. Then $a : b :: y : \frac{by}{a} = z$; and therefore $\frac{1}{2}yz = \frac{byy}{2a} =$ the Area of the flowing Triangle abc; and $\frac{byy\dot{x}}{2a} =$ the Fluxion of the Pyramid; again $c : a :: x : \frac{ax}{c} = y$, and $yy = \frac{a^2x^2}{c^2}$, therefore the Fluxion of the Pyramid is $\frac{ba^2x^2\dot{x}}{2ac^2} = \frac{abx^2\dot{x}}{2c^2}$, whoſe Fluent is $\frac{abx^3}{6c^2}$ $=$ ſolid Content of the Pyramid $abcD$. And when x becomes

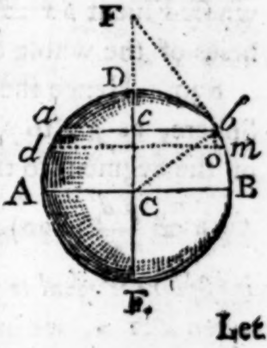
c,

c, or bD = BD, then $\frac{abx^3}{6c^3} = \frac{abc}{6} =$ the Solidity of the Pyramid ABCD. Hence every Priſm is equal to its Baſe $\frac{1}{2}ab$ multiplied by $\frac{1}{3}c =$ its Altitude.



834. To *cube the right* Cone AHB. Let AC $= a$, CH $= b$, and $p =$ Periphery of the Baſe; ſuppoſe the Section $adbe$ parallel to the Baſe ADBE; and make $ac = y$, cH $= x$; then will $a : p :: y : \frac{py}{a} =$ Periphery of the Circle $adbe$. And its Area will be $\frac{py}{a} \times \frac{1}{2}y$ (830) $= \frac{\frac{1}{2}pyy}{a}$; whence $\frac{\frac{1}{2}pyy\dot{x}}{a}$ will be the Fluxion of the Cone; and ſince $b : a :: x : \frac{ax}{b} = y$, we have $yy = \frac{a^2x^2}{bb}$; and the Fluxion above will become $\frac{\frac{1}{2}pa^2x^2\dot{x}}{ab^2} = \frac{\frac{1}{2}pax^2\dot{x}}{b^2}$; the Fluent of which is $\frac{pax^3}{6b^2}$; and when $x = b$, then the Fluent is $\frac{pab}{6} =$ the Solidity of the Cone AHB.

835. Becauſe $\frac{pab}{6} = \frac{1}{3}$ of $\frac{pab}{2}$, it appears that the Cone is *one Third of a Cylinder, having the ſame Baſe and Altitude.* (831) Alſo becauſe $\frac{pab}{6} = \frac{pa}{2} \times \frac{b}{3}$, it appears, *the Solidity of a Cone is equal to the Baſe multiplied into one third Part of its Height.*



836. *To cube the Sphere* ADBE. Let AC $= a$, and $p =$ Periphery of the Circle, whoſe Diameter is AB. Draw $ab \parallel$ AB, and put $ac = y$; then $a : p :: y = \frac{py}{a}$, the Periphery of a Circle on the Diameter ab; therefore $\frac{pyy}{2a} =$ Area of the ſaid Circle.

Let D $c = x$; then $\frac{pyy\dot{x}}{2a}$ is the Fluxion of the Segment a D b. But $yy = 2ax - xx$ (819), therefore $\frac{p\dot{x}}{2a} \times yy = \frac{p\dot{x}}{2a} \times 2ax - xx = \frac{2pax\dot{x} - pxx\dot{x}}{2a}$ = Fluxion of the Segment aDb; whoſe Fluent is $\frac{2pax^2}{4a} - \frac{px^3}{6a}$ = Solidity of the ſaid Segment. And when $x = 2a = d =$ D E, the Diameter, then the ſaid Fluent will become $\frac{pd^3}{2d} - \frac{pd^3}{3d} = \frac{pd^2}{6}$ = the Solidity of the Sphere.

837. Now ſince $\frac{pd^2}{6} = \frac{4a^2p}{6} = \frac{2}{3}a^2p = \frac{1}{3}a \times 2ap = \frac{1}{3}a \times dp$ it is evident, the *Solidity of a Sphere is equal to the Rectangle under the Diameter and Circumference of its greateſt Circle multiplied into* $\frac{1}{3}$ *of its Semidiameter.* Alſo ſince $\frac{pd^2}{6} = \frac{2}{3} \times \frac{pd^2}{4}$, and $\frac{pd}{4} \times d = \frac{1}{2}pa \times d$; *therefore every Sphere is equal to* $\frac{2}{3}$ *of its circumſcribing Cylinder* (831).

838. To *ſquare the Superficies of the Sphere*; draw the Tangent F b meeting the Axis produced in F, and join C b; draw $b\,o$ ∥ DE, and dm infinitely near, and parallel to ab; then are the Triangles C $b\,c$ and C F b ſimilar (659.) And ſo cb : Cb : : $ba : bm$, or $y : a :: \dot{x} = \frac{a\dot{x}}{y} = \dot{z} = bm$, the Fluxion of the Arch D b; which multiplied by $\frac{py}{a}$ (836,) gives $\frac{apy\dot{x}}{ay} = p\dot{x}$ = the Fluxion of the Superficies of the Segment a D b, whoſe Fluent px = Area thereof, and when $x = d$, the Superficies of the whole Sphere will be pd, as required.

839. Hence the Superficies of a Segment is to that of the Sphere, as px to pd, or as x to d, that is, as the Altitude Dc of the Segment to the Diameter D E of the Sphere. And ſince $\frac{1}{2}pa = \frac{pd}{4}$ (830) $= \frac{1}{4} \times pd$; it appears *the Superficies of the Sphere is equal to four times the Area of its greateſt Circle.* Hence, when $d = 1$, we have $p = 3{,}14159265$ (825).

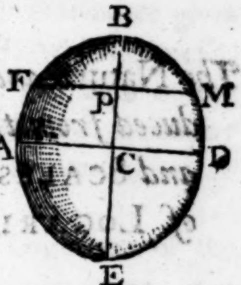
840.

840. If we put D, P, for the Diameter, and Periphery of one Circle, and *d*, *p*, for the ſame in another; and ſince D : *d* : : P : *p*; and the Areas being denoted by A and *a*, we have A : *a* : : $\frac{1}{4}$ DP : $\frac{1}{4}$ *dp* : : D P : *dp* : : DD : *dd*; *that is, the Areas of Circles are as the Squares of their Diameter.*

841. Let S and *s* denote the Solidity of two Spheres. Then S : *s* : : $\frac{1}{6}$ P D^2 : $\frac{1}{6}$ *p d*^{2} : : P D^2 : *p d*^{2} : : D^3 : *d*^{3}; (836) that is, *the Spheres are to each other as the Cubes of their Diameter.*

842. Let the Superficies of two Spheres be denoted by F and *f*. Then F : *f* : : P D : *p d* : : D^2 : *d*^{2}. (838) That is, *the Surfaces of Spheres are to each other as the Squares of their Diameters.*

843. From what has been ſaid, it follows, that D^3 : D P : : D^2 : P; that is, *any Cube* (D^3) *is to the Superficies of its inſcribed Sphere* (D P) *as a Square* (D^2) *is to its inſcribed Circle.*



844. To cube the ELLIPTIC SPHEROID ABDE. Here BP = *x*, PM = *y*, BE = *a*, AD = *b*, *p* = Parameter. Then $yy = px - \frac{px^2}{a}$ (763.) Let the Radius of a Circle be to the Circumference, as *r* : *c*; then *r* : *c* : : *y* : $\frac{cy}{r}$ = Periphery of the Circle, whoſe Diameter is F M; then $\frac{cyy}{2r}$ = Area of the Circle; and ſo $\frac{cyy\dot{x}}{2r} = \frac{pc\dot{x}}{2r} - \frac{pcx^2\dot{x}}{2ra}$ = Fluxion of the Segment F B M; whoſe Fluent $\frac{pcx^2}{4r} - \frac{pcx^3}{6ra}$ = Solidity thereof. When $x = a$, then the ſaid Fluent is $\frac{pca^2}{4r} - \frac{pca^3}{6ra}$; $= \frac{pca^2}{12r}$ = Solidity of the *Spheroid*, as required.

845. Now becauſe *a* : *b* : : *b* : $p = \frac{bb}{a}$ (766) and $b = 2r$; we have $\frac{pca^2}{12r} = \frac{cab}{6}$ = the Solidity of the Spheroid. And becauſe

$\frac{cba}{6} = \frac{cb}{6} \times a$, and $\frac{cb}{4} \times a$ = Cylinder circumſcribed about the Spheroid, 'tis evident, *the Spheroid is alſo* $\frac{2}{3}$ *of its circumſcribing Cylinder.*

846. Hence in the Sphere $\left(\frac{cd^2}{6}\right)$ and Spheroid $\left(\frac{abc}{6}\right)$, when $d = b$, it is the Sphere : Spheroid :: $cd^2 : abc :: d : a ::$ Conjugate : Tranſverſe Axis of the Ellipſoid.

$$\text{847. Hence the Solidity of } \left\{\begin{array}{l} \text{the Cube} = 10000. \\ \text{the Cylinder} = 7854. \\ \text{the Sphere} = 5236. \\ \text{the Cone} = 2618. \end{array}\right.$$

CHAP. XV.

The Nature *and* Geneſis *of* LOGARITHMS, *deduced from the* HYPERBOLA, *the* MODULES *and* SCALES *of* LOGARITHMS; *the* FLUXIONS *of* LOGARITHMS, *and their Uſe.*

848. I Have already obſerved, that a Logarithm is only an Expreſſion of the Place or Diſtance from Unity, which any given Number holds in a Series of geometrical Proportionals (ſee 147). I have alſo ſhewn how thoſe Logarithms are inveſtigated and expreſſed by *Numbers.* Alſo from the Nature and Quadrature of the *Hyperbola* it appears (784, 785, 786) that the Ratio of any Numbers to Unity, may alſo be expreſſed by hyperbolic Sectors or Spaces, and conſequently they are alſo properly the Logarithms of ſuch Numbers. And we ſhall now more fully proſecute and explain this uſeful Doctrine.

849. It has been ſhewn, that the Aſymptotic Space ADVB $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. is the Logarithm of the Ratio of CD to CA, that is, of $a + x$ to a, or of $1 + x$ to 1 (786). And becauſe $\frac{\dot{x}}{1 + x}$ is the Fluxion of this Ratio, therefore it appears, *that the Fluxion of the Logarithm of any Number,*

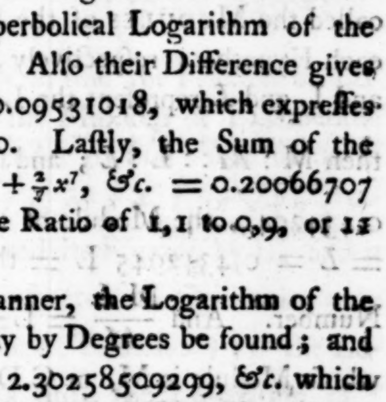
$1 + x$

$1 + x$, *is equal to the Fluxion* ($\dot{x}$) *of that Number divided by the Number itſelf.*

850. If we take any Number leſs than Unity, as C*A*; (See Fig. to Art. 829.) then *A* D $= x$, and the Logarithm of CD : C*A*, or of $1 : 1 - x$, is the Space *A B* V D $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$, *&c.* Whence the Logarithm of the Ratio of C A : C *A*, or of $1 \pm x$ to $1 - x$, the Space A B *B A* $= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7$, *&c.* = Sum of the other two Series, (Art. 849.)

851. Let A D $=$ *A* D $= \frac{1}{10} = x$, then A C $= 1 + \frac{1}{10}$, and *A* C $= 1 - \frac{1}{10}$. Now half the Sum of the above two Series is $x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7$, *&c.* and half their Difference is $\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{6}x^6 + \frac{1}{8}x^8 +$, *&c.* which computed in Numbers make, 0,100335347731 0; and 0,00502516792 67; the *Sum* of theſe gives the greater Logarithm *A B* V D = 0.1053605156, which is the hyperbolical Logarithm of the Ratio of 1 to 0.9, or of 10 to 9. Alſo their Difference gives the leſſer Logarithm A B V D = 0.09531018, which expreſſes the Ratio of 1.1 to 1, or 11 to 10. Laſtly, the Sum of the two Series, *viz.* $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7$, *&c.* = 0.20066707 = *A B* B A, the Logarithm of the Ratio of 1,1 to 0,9, or 11 to 9.

852. By proceeding in this Manner, the Logarithm of the Ratio of any Number to Unity may by Degrees be found; and that of 10 to 1 will be found to be 2.30258509299, *&c.* which Number is therefore called the *hyberbolical Logarithm* of 10. But in the Logarithms of the Tables in common Uſe, the Logarithm of the ſame Ratio of 10 to 1 is 1,0000000, *&c.* which is called the *Tabular Logarithm* of 10.



853. Let there be two ſimilar Hyperbolas BVG and *BVG*, *viz.* ſuch *whoſe tranſverſe and conjugalate Diameters are proportional*, or C V : C *V* : : C D : C *D*. And let C A : C D, C *A* : C *D* in each be the Ratio of 10 to 1. And let the Logarithm of this Ratio in the firſt, *viz.* A D V B = 2,3025851, *&c.* and the Logarithm of the ſame Ratio in the latter, *viz.* *ADVB* = 1,0000000. Then it

will be A D V B : *A D V B* : : C D V E : *C D V E*. For put C D $= a$, A D $= x$; C *D* $= \mathrm{a}$, *A D* $= \mathrm{x}$; then will A D V B $= a^2 \times \overline{\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3}, \&c.}$ And *A D V B* $= \mathrm{a}^2 \times \overline{\frac{\mathrm{x}}{\mathrm{a}} + \frac{\mathrm{x}^2}{2\mathrm{a}^2} - \frac{\mathrm{x}^3}{3\mathrm{a}^3}, \&c.}$ Now ſince C A : C D : : *C A* : *C D*, that is, $a : x :: \mathrm{a} : \mathrm{x}$; we have $a\mathrm{x} = \mathrm{a}x$; and ſo $\frac{x}{a} = \frac{\mathrm{x}}{\mathrm{a}}$; therefore $\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3}$, &c. $= \frac{\mathrm{x}}{\mathrm{a}} - \frac{\mathrm{x}^2}{2\mathrm{a}^2} - \frac{\mathrm{x}^3}{3\mathrm{a}^3}$, &c. and conſequently it is A D V B : *A D V B* : : $a^2 : \mathrm{a}^2$: : *C D V E* : *C D V E* : 2,3025851 : 1.0000000 : : 1 : 0.4342945.

854. Now theſe Parallelograms C D V E, *C D V E*, are called the MODULES of the *Syſtems* or *Scales* of Logarithms, in each Hyperbola reſpectively; let theſe be called M and *M*; and L and *L* repreſent the Logarithms of any given Number; then M : *M* : : L : *L*; and ſo $\frac{M\,\mathrm{L}}{\mathrm{M}} = L$, put $\frac{M}{\mathrm{M}} = \mathrm{R} =$ 0,4342945, the Module of the common Tables. Then R L $= L =$ 0,4342945 L $=$ the *tabular Logarithm* of the given Number. And $\frac{\mathrm{M}\,L}{M} = \mathrm{L} =$ 2,3025851 *L* $=$ the Logarithm whoſe Module is M, or C D V E.

855. To illuſtrate this, let a Series of Numbers, and their Logarithms be as follow:

0	1	2	3	4	5	6, &c.	Logarithms.
1.	2.	4.	8.	16.	32.	64.	Numbers.

Here ſince the Logarithms are given, we can find the Module; for L : M : : *L* : *M*. Thus the Logarithm of 8 in this Series is 3, in the Tables it is 90309, therefore 0,90309 : 0.4342944819 : : 3 : 1,4427 $=$ *M*, the Module of the Logarithms in the Series above. And having the Module of the Logarithm or Ratio, 'tis eaſy to find the Logarithm by the Reverſe of the laſt Analogy, *viz.* M : L : : *M* : *L*. Thus 0.4343 : 0.90309 : : 1.4427 : 3 $=$ the Logarithm of 8.

856. Alſo the hyperbolical Logarithms are eaſily produced by multiplying thoſe of the common Tables by 2,3025851, as in the following Examples of all the Numbers from 1 to 10. Thus

Table of Logarithms.		Hyperb. Logarithms.
1—0.0000000	Multipled by 2.3025851 produce.	1—0.0000000
2—0.3010300		2—0.6931472
3—0.4771213		3—1.0986123
4—0.6020600		4—1.3862943
5—0.6989700		5—1.6094379
6—0.7781513		6—1.7917594
7—0.8450980		7—1.9459101
8—0.9030900		8—2.0794415
9—0.9542425		9—2.1972241
10—1.0000000		10—2.3025851

857. We ſhall next ſhew how to find the Fluxion of Logarithms and the Powers; as alſo the Fluxions of Exponential Quantities, of any Degree. In order to find the Fluxion of the Logarithm of any Quantity, as x, or $x + 1$, or $\overline{x+a}^m$, we ſhall denote the Logarithm by L, and its Fluxion by $\dot{L}$; and it is to be underſtood that L, and $\dot{L}$ reſpects thoſe Quantities only to which they are immediately prefixed: Thus $L\,\overline{x+y}$ is the Logarithm of $\overline{x+y}$; but $L\,x+y$ is only the Logarithm of x added to the Quantity y.

858. Hence it will be eaſy to find the Fluxion of any Kind of Logarithm. For Example; $\dot{L}\,\overline{xx+yy} = \dfrac{2x\dot{x} + 2y\dot{y}}{xx+yy}$. $\dot{L}\,\overline{ax^2+x^3} = \dfrac{2ax\dot{x} + 3x^2\dot{x}}{ax^2+x^3} = \dfrac{2a\dot{x} + 3x\dot{x}}{ax+x^2}$; and univerſally $\dot{L}\,\overline{z^m+x^n}^q = \dfrac{qmz^{m-1}\dot{z} + q^n x^{n-1}\dot{x}}{z^m+x^n}$. For put $z^m + x^n = y$, then will $\overline{z^m+x^n}^q = y^q$; and conſequently the Fluxion of the Logarithm thereof will be $= q\dot{y} \times \dfrac{y^{q-1}}{y^q} = q\dot{y} \times \dfrac{1}{y} = \dfrac{q\dot{y}}{y}$; which will be found equal to the Fluxion above, by ſubſtituting the Values of y, and $\dot{y}$.

859.

859. The Fluxions of the Powers of Logarithms are found by the general Rules for the Fluxion of the Powers of flowing Quantities. Thus admit the Fluxion of the 2d, or *Square Power* of the Logarithm of x were required, or Fluxion of $\overline{L x}^2$; we muſt firſt multiply by the Index of the Power, *viz.* 2, which makes $2\,\overline{L x}^2$; then is the Index to be leſſened by Unity, which makes $2\,\overline{L x}^{2-1} = 2\,L x$; laſtly, we muſt multiply this into the Fluxion of the Root $L x$, which is $\frac{\dot{x}}{x}$, and it will produce $2\,L x \times \frac{\dot{x}}{x}$ the Fluxion of $L^2 x$, as required.

860. And in general the Fluxion of $L^m x = m L^{m-1} x \times \frac{\dot{x}}{x}$; and $L^n \overline{x+a} = n L^{n-1} \overline{x+a} \times \frac{\dot{x}}{x+a}$; and $L^m \overline{x+a}^n = m L^{m-1} \overline{x+a}^n \times \frac{n\dot{x}}{x+a}$. For $L^m \overline{x+a}^n$ multiplied by the Index of the Power is $m L^m \overline{x+a}^n$; and that with its Exponent m leſſened by Unity, is $m L^{m-1} \overline{x+a}^n$, and this being multiplied by the Fluxion of the Root $L \overline{x+a}^n$, (which is $\frac{n\overline{x+a}^{n-1}\dot{x}}{\overline{x+a}^n} = \frac{n\dot{x}}{x+a}$,) will make $m L^{m-1} \overline{x+a}^n \times \frac{n\dot{x}}{x+a}$ the ſame as above. And thus may the Fluxions of the Powers of Logarithms, however compounded with other Quantities, or Powers of Quantities, be found.

861. From what has been ſaid, we may obſerve that the Fluxions of the Logarithms of the ſeveral Powers of any Quantity, as x, are in the ſame Ratio as the Exponents of thoſe Powers, and conſequently as the Logarithms themſelves. Thus $\dot{L} x^m$ and $\dot{L} x^n$ are $\frac{m\dot{x}x^{m-1}}{x^m} = \frac{m\dot{x}}{x}$ and $\frac{n\dot{x}x^{n-1}}{x^n} = \frac{n\dot{x}}{x}$; but $\frac{m\dot{x}}{x} : \frac{n\dot{x}}{x} :: m : n$. Which proves the Truth of what I ſaid.

862. We have hitherto ſuppoſed the Indexes, or Exponents of Powers *determinate*, or *invariable*, but in ſome Caſes we ſhall find, that they alſo are variable or flowing Quantities, and have

accordingly their Indexes or Exponents. Now the Fluxions of thoſe Exponential Quantities, whoſe Indexes are variable, are found by Means of the Logarithms very eaſily, as in the following Examples.

863. Let it be required to find the Fluxion of a^x, where a is a conſtant Quantity, and its Index x a flowing one: Put $a^x = z$, then will $\dot{z}$ be equal to the Fluxion of a^x. Becauſe $a^x = z$, if we take the Logarithms on both Sides, we ſhall have $\mathrm{L}\,a^x = \mathrm{L}\,z$; and *ſince the Logarithm of the Root of any Power multiplied by the Index of the Power, is the Logarithm of that Power*, therefore $x\,\mathrm{L}\,a = \mathrm{L}\,a^x$, and ſo $x\,\mathrm{L}\,a = \mathrm{L}\,z$; and taking the Fluxions, we have $\dot{x}\,\mathrm{L}\,a = \frac{\dot{z}}{z}$, and therefore $z\,\dot{x}\,\mathrm{L}\,a = a^x\,\mathrm{L}\,a\,\dot{x} = \dot{z}$; and conſequently $a^x\,\mathrm{L}\,a\,\dot{x}$ is the Fluxion of a^x, as required.

864. Again, ſuppoſe it required to find the Fluxion of z^x, where both z and x are flowing Quantities, and make $z^x = v$, then will $\mathrm{L}\,z^x = \mathrm{L}\,v = x\,\mathrm{L}\,z$. The Fluxions of theſe will be $\frac{\dot{v}}{v} = \dot{x}\,\mathrm{L}\,z + x\frac{\dot{z}}{z} = \frac{\dot{v}}{z^x}$ (becauſe $v = z^x$,) and by reducing the Equation, we have $z^x\,\dot{x}\,\mathrm{L}\,z + z^{x-1}\,x\,\dot{z} = \dot{v}$; which therefore is the Fluxion ſought. *Note*, $x\frac{\dot{z}}{z} = x\dot{z} \times \frac{1}{z}$, and $\frac{1}{z} \times z^x = \frac{z^x}{z} = z^{x-1}$; *and therefore* $x\frac{\dot{z}}{z} \times z^x = z^{x-1}\,x\,\dot{z}$, *as in the latter Part of the Fluxion.*

865. Laſtly, let the Quantity z^{y^x} be propoſed, to find its Fluxion. Let $z^{y^x} = v$; then $\mathrm{L}\,z^{y^x} = \mathrm{L}\,v$, or $y^x\,\mathrm{L}\,z = \mathrm{L}\,v$; and taking the Fluxions, we have the Fluxion of $y^x = y^x\,\mathrm{L}\,y\,\dot{x} + y^{x-1}\,x\,\dot{y}$ (864) which muſt be multiplied into $\mathrm{L}\,z$, which makes $\mathrm{L}\,z \times y^x\,\mathrm{L}\,y\,\dot{x} + \mathrm{L}\,z \times y^{x-1}\,x\,\dot{y}$; to which muſt be added the Fluxion of $\mathrm{L}\,z \times y^x$, or $y^x\frac{\dot{z}}{z}$; and the Sum will be equal to the Fluxion of $\mathrm{L}\,v$, or $\frac{\dot{v}}{v}$; thus $\mathrm{L}\,z\,y^x\,\mathrm{L}\,y\,\dot{x} + \mathrm{L}\,z\,y^{x-1}\,x\,\dot{y} + y^x\frac{\dot{z}}{z} = \frac{\dot{v}}{v} = \frac{\dot{v}}{z^{y^x}}$. And therefore multiplying by z^{y^x}, we have

$z^{y^x} Lz \times y^x Ly\,\dot{x} + z^{y^x} Lz \times y^{x-1} x\dot{y} + y^x \dot{z}\, z^{y^x-1} = \dot{v}$; and is therefore the Fluxion of the Expreſſion z^{y^x}, as required.

CHAP. XVI.

The METHOD *of finding* FLUENTS *by the Meaſures of* RATIOS *and* ANGLES, *by the Help of* TABLES *of* LOGARITHMS; *and of Natural* SINES, TANGENTS, *&c.*

866. AS the Method of finding *Fluents* of given Fluxions is the moſt neceſſary, and yet the moſt abſtruſe Part of the *new Matheſis*, it is no Wonder if it has exerciſed the inventive Faculty of the greateſt Geniuſes of the laſt and preſent Age. And ſince in the *Geometry* of *Curve Lines*, and eſpecially in the Solution of *Phyſical Problems*, (which are of the higheſt Importance in the Sciences), Caſes will often occur, which make it neceſſary to have Recourſe to *Infinite Series*, for procuring the Fluents required, and this often proves a laborious and diſagreeable Taſk; and which of Courſe the Student would be glad to avoid or exchange for ſome other more eaſy and practical Method.

867. Now there are ſeveral Forms of Fluxions whoſe Fluents may be readily obtained in the Meaſure of *a Ratio*, and alſo of the *Arch of a Circle*; and conſequently, as all Ratios of Numbers to Unity, are already calculated in the common Tables of Logarithms of Numbers; and any Arch of a Circle is known from its *Right Sine*, *Verſed Sine*, *Tangent*, or *Secant*, calculated in the Trigonometrical Canon, it follows, thoſe Tables will greatly facilitate the Buſineſs of finding Fluents; and the Method of uſing them for that Purpoſe, we ſhall next explain and illuſtrate ſomewhat farther than has been yet done for the Capacity of a *Learner*.

868. For this Purpoſe he muſt recollect, *That the Fluxion of a Hyperbolic Logarithm of any Number, is ever expreſſed by the Fluxion of that Number, divided by the Number itſelf* (849). And

hence will ariſe the four following *Logarithmic* Forms of Fluxions, whoſe Fluents may be had from a Table of Logarithms, which would otherwiſe require an *infinite Series* to determine them, *viz.*

869. The Fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expreſſed by the hyperbolical Logarithm of $x + \sqrt{x^2 \pm a^2}$: For the Fluxion of $(x + \sqrt{x^2 \pm a^2})$ the Number itſelf, being $\dot{x} + \frac{x\dot{x}}{\sqrt{x^2 \pm a^2}}$ $= \frac{\dot{x}\sqrt{x^2 \pm a^2} + x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}} \times \overline{\sqrt{x^2 \pm a^2} + x}$, this laſt Quantity, divided by that Number, gives $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, the very Fluxion firſt propoſed; which is the *firſt Form*.

870. It alſo appears that the Fluent of $\frac{\dot{x}}{\sqrt{2ax + x^2}}$ will be truly expounded by the hyperbolical Logarithm of $a + x + \sqrt{2ax + x^2}$: Becauſe the Fluxion of the Number $(a + x + \sqrt{2ax + x^2})$ is here $= \dot{x} + \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}} = \frac{\dot{x}}{\sqrt{2ax + xx}}$ $\times \overline{\sqrt{2ax + xx} + a + x}$; which divided by that Number produces $\frac{\dot{x}}{\sqrt{2ax + xx}}$; which is the *ſecond Form*.

871. Likewiſe the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ will be repreſented by the hyperbolical Logarithm of $\frac{a + x}{a - x}$: Becauſe the Fluxion of $\frac{a + x}{a - x}$, being $\frac{\dot{x} \times \overline{a - x} + \dot{x} \times \overline{a + x}}{\overline{a - x}|^2} = \frac{2a\dot{x}}{\overline{a - x}|^2}$, if the ſame be therefore divided by $\frac{a + x}{a - x}$, we ſhall have $\frac{2a\dot{x}}{\overline{a - x}|^2} \times \frac{a - x}{a + x}$ $= \frac{2a\dot{x}}{\overline{a - x} \times \overline{a + x}} = \frac{2a\dot{x}}{a^2 - x^2}$; which is the *third Form*.

872. Laſtly, the Fluent of $\frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$ will be denoted by

the hyperbolical Logarithm of $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$; for here the Fluxion of the Number is $\frac{\mp x\dot{x}}{\sqrt{a^2\pm x^2}}\times\frac{\dot{\overline{a+\sqrt{a^2\pm x^2}}}}{\overline{a+\sqrt{a^2\pm x^2}}|^2}$

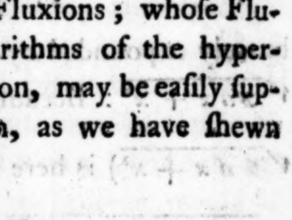
$$\mp\frac{x\dot{x}}{\sqrt{a^2\pm x^2}}\times\frac{\dot{\overline{a-\sqrt{a^2\pm x^2}}}}{\overline{a+\sqrt{a^2\pm x^2}}|^2}=\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2};$$

which divided by $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2}$

$$\times\frac{a+\sqrt{a^2\pm x^2}}{a-\sqrt{a^2\pm x^2}}=\frac{\pm 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}\times\overline{a-\sqrt{a^2\pm x^2}}}$$

$=\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\mp x^2}=\frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$, the Fluxion propoſed, whch is the *fourth Form.*

Theſe four are the principal Forms of Fluxions; whoſe Fluents may be found from a Table of Logarithms of the hyperbolic Kind: Which Table, upon Occaſion, may be eaſily ſupplied by a Table of the common Form, as we have ſhewn (856).



873. Four other Forms of Fluxions ſimilar to theſe, expreſs the Fluxion of the Arch of a Circle in Terms of the Sine, Verſed Sine, Tangent and Secant of that Arch. For let A D be the Arch of a Circle, whoſe Center is C, its Sine B D, Verſed Sine A B, Tangent A E, and Secant C E, let *b d*, be drawn indefiniently near to B D, and draw D *a*, parallel to A C, then ſince the very ſmall Arch D *d*, may be eſteemed a right Line, the ſmall fluxionary Triangle D *a d* will be ſimilar to the Triangle D B C or A E C, for the Angle at *a*, and B is a right one in each; alſo, if from the Right Angles C D *d*, and B D *a*, we take the common Angle C D *a*, there will remain the Angle *d* D *a* = B D C, and the Sides in each, *viz.* *d a*, and D *a* being parallel to the Sides B D and B C, they will be proportional, and the Triangles ſimilar by (621, 657).

874.

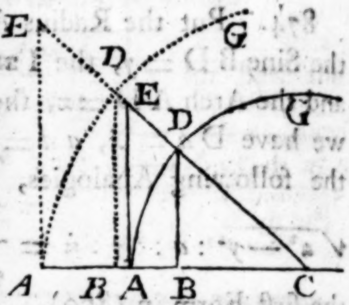
874. Put the Radius A C $= a$, the Verſed Sine A B $= x$, the Sine B D $= y$, the Tangent A E $= t$, the Secant C E $= s$, and the Arch A D $= z$, then in the fluxionary Triangle D $a\,d$, we have D $a = \dot{x}$, $a\,d = \dot{y}$, and D $d = \dot{z}$, whence we have the following Analogies, C B : C D : : D a : D d, that is, $\sqrt{a^2 - y^2} : a :: \dot{y} : \dot{z} = \dfrac{a\dot{y}}{\sqrt{a^2 - y^2}}$, which correſponds to the firſt Form in (869).

875. Again, we have the following Analogy in Terms of the Verſed Sine, as B D : C D : : D a : D d, that is, $y : a :: \dot{x} : \dot{z} = \dfrac{a\dot{x}}{y} = \dfrac{a\dot{x}}{\sqrt{2ax - xx}}$ by (636). This Form is analogous to the ſecond Form in (870.)

876. The Fluxion of the ſame Arch, expreſſed in Terms of the Tangent, we have already ſhewn in (823,) is $\dot{x} = \dfrac{a^2\dot{t}}{a^2 + t^2}$, which is ſimilar to the third Form in (871).

877. Laſtly, by ſimilar Triangles CBD, CAE, we have CE : C A : : C D : C B; that is, $s : a :: a : \dfrac{a^2}{s}$; hence A B $= x = a - \dfrac{a^2}{s}$, whoſe Fluxion is $\dot{x} = \dfrac{a^2\dot{s}}{s^2}$; but A E : C E : : D a : D d; that is, $\sqrt{s^2 - a^2} : s :: \dfrac{a^2\dot{s}}{s^2} : \dot{z} = \dfrac{a^2\dot{s}}{s\sqrt{s^2 - a^2}}$, which is like the fourth Form in (872).

878. Theſe Forms differ in nothing from the former, but the Signs and conſtant Quantities, by which it will be eaſy to know when the Fluent is to be ſought for in the Meaſure of a Ratio, or of an Angle; that is from a *Table of Logarithms*, or the Trigonometrical *Canon of Sines*, *Tangents*, &c. But as in the above Forms, the Radius is expreſſed by (a), and the Radius of the Table by Unity, or 1.000000; therefore to accomodate them to the Tables in uſe, we take the Expreſſion of the *Sine*, *Tangent*, &c. ſuch as pertain to a Circle, whoſe Radius $= 1$, in the following Manner.



879. With the Radius C*A* = *a*, deſcribe the Circle *A D G*; and with the Radius C A = 1, deſcribe the Circle A D G; draw the indefinite Line C*E*, and from the Points D, *D*, let fall the Perpendiculars D B, *D B*; and at A and *A* raiſe the Perpendiculars AE, *AE*. Then by ſimilar Triangles C A : C*A* :: AD : *AD*; whence (putting AD = D, and *A D* = A), we have 1 : *a* :: D : *a*D = A. Again, put D B = y, and *DB* = *y*; then C D : C*D* :: D B : *D B*; that is, 1 : *a* :: y : *a* y = *y*. In like Manner, by making A E = t, and *AE* = *t*; we have 1 : *a* :: t : *a* t = *t*; and if C E = s, and C*E* = *s*; then 1 : *a* :: s : *a* s = *s*. Laſtly, putting A B = x, and *AB* = *x*, then B C = 1 — x, and *B* C = *a* — *x*, but it is 1 : *a* :: 1 — x : *a* — *x*, and conſequently 1 : *a* :: x : *x* = *a* x.

880. Hence the Tabular Sines, Tangents, &c. expreſſed in Terms of the above fluxionary Forms, will be $\mathrm{y} = \frac{y}{a}$, $\mathrm{t} = \frac{t}{a}$, $\mathrm{s} = \frac{s}{a}$, and $\mathrm{x} = \frac{x}{a}$. And therefore ſince the Fluxion of the Arch *AD* (= A = *a* × D) is $\frac{a\dot{y}}{\sqrt{a^2 - y^2}}$ (874); the Fluxion of the Arch A D = D, will be only $\frac{\dot{y}}{\sqrt{a^2 - y^2}}$; and conſequently it will follow,

$$\text{That the Fluent of} \left\{ \begin{array}{c} \dfrac{\dot{y}}{\sqrt{a^2 - y^2}} \\[2ex] \dfrac{\dot{x}}{\sqrt{2ax - x^2}} \\[2ex] \dfrac{a\dot{t}}{a^2 + t^2} \\[2ex] \dfrac{a\dot{s}}{s\sqrt{s^2 - a^2}} \end{array} \right\} \begin{array}{c}\text{is equal to the Arch, whoſe} \\ \text{Radius is Unity, and}\end{array} \left\{ \begin{array}{l} \text{Right Sine} \\[2ex] \text{Verſed Sine} \\[2ex] \text{Tangent} \\[2ex] \text{Secant} \end{array} \right\} \text{is} \left\{ \begin{array}{c} \dfrac{y}{a} \\[2ex] \dfrac{x}{a} \\[2ex] \dfrac{t}{a} \\[2ex] \dfrac{s}{a} \end{array} \right.$$

881. But to apply this Doctrine more generally to Uſe, with regard to finding the Fluent by Logarithms or Trigono-

metrical Numbers, it is to be obſerved, that if the Terms of the Ratio conſiſt of ſeveral Quantities, as $\frac{A+B}{C}$, and any one of them, as A, be conſtant or given, and the Relation of the Reſt depend upon their Connection with it, then the ſaid Quantity A may be aſſumed for the *Module* of that Ratio; and then (according to the Method invented by the late learned Mr. *Cotes*) it may be expreſſed in this Manner, $A\left|\frac{A+B}{C}\right.$; *viz. the Ratio of* A + B *to* C, *in that Scale of Proportionals, the Module of whoſe Logarithms is* A.

882. Thus, ſuppoſe it were required to find the Fluent of the Fluxion $\frac{6a\dot{x}}{b\times\overline{a^2-x^2}}$; this you will obſerve conſiſts of a conſtant and a fluxionary Part, *viz.* $\frac{3}{b}\times\frac{2a\dot{x}}{a^2-x^2}$, and the Fluent of the Latter is $\frac{a+x}{a-x}$ (871), which, (as it is a Ratio, and the Part (a) invariable), may be thus expreſſed, $a\left|\frac{a+x}{a-x}\right.$; the Logarithm, or Meaſure of this Ratio is a L R *, which is the Fluent of $\frac{2a\dot{x}}{a^2-x^2}$; therefore $\frac{3a}{b}\left|\frac{a+x}{a-x}\right.$, or $\frac{3a}{b}\times$ L R, is the Fluent of $\frac{6a\dot{x}}{b\times\overline{a^2-x^2}}$, as required.

883, To purſue the *Coteſian Method* yet farther, let the Right angled Triangle ACE be propoſed, See Fig. to (873) make AC = R = *Radius*; then will AE = T = *Tangent*, and CE = S = *Secant*, of the Angle ACE. Then if it be propoſed to find the Meaſure of the Ratio of the Sum of the Legs to the Hypothenuſe, the Baſe AC being the Module, the Expreſſion will be thus; $R \left| \frac{R+T}{S} \right.$, or if $R = a$, $T = y$, then will $S = \sqrt{aa + yy}$; and the Ratio ſtands thus; $a \left| \frac{a+y}{\sqrt{aa+yy}} \right.$. If now $R = 4$, $T = 3$, $S = 5$; then will the Ratio in Numbers be $4 \left| \frac{4+3}{5} \right. = 4 \left| \frac{7}{5} \right.$; the Meaſure of which is found as before, thus; 0.4343 : 0.14613 (= Table of Logarithms of $\frac{7}{5}$) : : 4 : 2.6917 = Logarithm or Meaſure of the Ratio of $\frac{7}{5}$ to the Modulus 4.

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* *N. B.* A Miſtake in Art. 854, paſſed (thro' Inadvertency) uncorrected.—The latter Part of that Article is to be read thus; M : *M* :: (0.4342945 : 1 ::) L : *L*; and ſo $\frac{M\,L}{\mathrm{M}}=L$; put $\frac{M}{\mathrm{M}}\left(=\frac{1}{0{,}4342945}\right)$ = R = 2,3025851; then will R L = *L* = *Hyperbolical Logarithm*, or Meaſure of a Ratio in the Scale, whoſe Module is 1. Alſo *M* and *L* may repreſent the Module and Logarithm of any other Scale beſides that of the Hyperbola, and accordingly R may be determined.

884. If the Fluent cannot be found in the Expreſſion of a Ratio, (as it ſometimes happens) then we can often obtain it by the Meaſure of an Arch of a Circle, as A D, (See Fig. to 873), or the given Angle A C E; for ſince the Circumference of a Circle is to its Diameter, as 3.14159, *&c.* to 1, if we ſay, as 3.14159, *&c.* : 1 : : 180 Degrees : 57.2957795, *&c.* = 57° 17′ 44″ = the Degrees in the Arch of a Circle equal to Radius; this Number therefore is called the *Trigonometrical Modulus*, and is repreſented by K = 57.2957795, *&c.* and if $\frac{1}{57.2957795} = 0.01745329252 = \frac{1}{K} = k$, then is k called the *Reciprocal Modulus*.

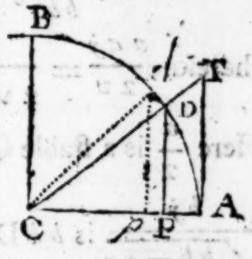
885. Suppoſe now any Arch of a Circle A D = D; then it will be, *As the Radius* A C *in Degrees to the Arch* A D *in Degrees, ſo is the Radius* A C *in Numbers to the Arch* A D *in Numbers*; that is, as $K : D : : R : \frac{R}{K} D = R k D$ = the Arch A D in Numbers. For Example, If $R = 1$, and A D = D = 30°, then $R k D = 1 \times 0.0174530 \times 30 = 0.523598$, *&c.* = the Meaſure of the Arch A D in Numbers.

886. In order to find the Fluents in the Meaſure of a Ratio or Angle, in the Manner as has been taught, Mr. *Cotes* invented a great many *general Forms of fluxionary Expreſſions* (to which others are to be reduced) and diſpoſed in Tables, the various Fluents pertaining thereto, according to the different Values of the Exponent of the flowing Quantity, affirmative or negative; which the Reader will find at large in his learned Treatiſe, entituled *Harmonia Menſurarum*.

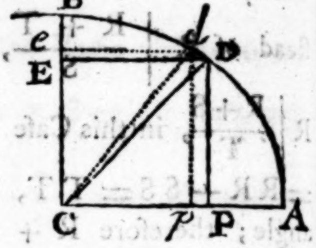
887.

887. It will be neceſſary to obſerve here, that, if inſtead of $R\,\Big|\,\frac{R+T}{S}$, we ſhould meet with the Ratio $R\,\Big|\,\frac{R+S}{T}$, in this Caſe $\frac{R+S}{T} = \frac{T}{S-R}$; for $\overline{R+S}\times\overline{R-S} = RR - SS = TT$, by the Property of a right angled Triangle; therefore $R + S : T :: T : R - S$; and conſequently $\frac{R+S}{T} = \frac{T}{S-R}$; and ſo $R\,\Big|\,\frac{R+S}{T} = R\,\Big|\,\frac{T}{R-S}$; which to know will be of uſe hereafter.

888. Alſo it will be proper to obſerve, that the negative Expreſſion of the Logarithm of a Ratio, as $-R\,\Big|\,\frac{R+T}{S}$, is made Affirmative by inverting the Ratio, as thus $+R\,\Big|\,\frac{S}{R+T}$. For $\frac{a^2}{a^5} = a^{-3}$; but $\frac{a^5}{a^2} = a^3$. Or if $-\frac{a}{b} = s$, then $-s = +\frac{b}{a}$. I ſhall illuſtrate this Doctrine by a Solution of the following Problems.



889. *To ſquare the* Sector *of a Circle* CAD. Let $CA = a$, $AP = x$, and $PM = y$, then by (874), we find the Fluxion of the Arch $AD = \frac{a\dot{y}}{\sqrt{aa-yy}} = Dd$; which multiplied by $\frac{a}{2}$ is $\frac{aa\dot{y}}{2\sqrt{aa-yy}} = d\,CD$, which ſmall Sector is the Fluxion of the Sector ACD. Now $\frac{a}{2}$ is a ſtable Quantity, and the other Part $\frac{a\dot{y}}{\sqrt{aa-yy}}$ being the Fluxion of the Arch AM, its Fluent will be found in the Meaſure of the Angle ACD, *viz.* it will be $= a\,k\,D$ (885); and if $AD = D = 30°$, the Length $AD = 0.523598$, &c. which multiplied by $\frac{a}{2}$, will give 0.261799, &c. = Area of the Sector ACD, when $a = 1$. (See Art. 828.)



890. *To ſquare the* Sector *of an Ellipſis* A C D. Let AC $= a$, CB $= b$, AP $= x$, CP $=$ ED $= v$, PD $= y$. 'Tis evident the Fluxion of the Space C A D E is E $d = v\dot{y}$; alſo the Triangle CDE $= \frac{vy}{2}$, whoſe Fluxion is $\frac{v\dot{y} + y\dot{v}}{2}$, which ſubtracted from the Fluxion $v\dot{y}$, leaves $\frac{\dot{y}v - \dot{v}y}{2} =$ to D C d, the Fluxion of the Sector A C D. And ſince from the Equation of the Curve $\frac{aa}{bb}yy = aa - vv$, we have in Fluxions, $\frac{aa}{bb}\dot{y}\dot{y} = - v\dot{v}$, or $\frac{aa}{bbv}y\dot{y} = -\dot{v}$; if we ſubſtitute this Value of $\dot{v}$ in the Expreſſion $\frac{\dot{y}v - \dot{v}y}{2}$, we ſhall have $\frac{\dot{y}}{2v} \times \overline{vv + \frac{aa}{bb}yy} = \frac{aa\dot{y}}{2v}$, becauſe $vv + \frac{aa}{bb}yy = aa$; alſo, becauſe it is $v = \frac{a}{b}\sqrt{bb - yy}$, therefore $\frac{aa\dot{y}}{2v} = \frac{ab\dot{y}}{2\sqrt{bb - yy}} =$ Fluxion of the Sector C A D. Here $\frac{a}{2}$ is a ſtable Quantity, and the Fluent of the other Part $\frac{b\dot{y}}{\sqrt{bb - yy}}$ is bkD, as ſhewn in the laſt Article; therefore $\frac{ab}{2}k$D $=$ Area of the Sector A C D, as required.

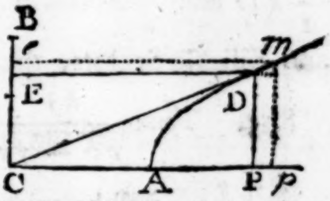
891. *To ſquare the* Sector *of an Hyperbola* A C D. By the very ſame Method as in the Ellipſis, we get the Fluxion $\frac{ab\dot{y}}{2\sqrt{bb + y}}$, for the hyperbolical Sector A C D, which differs only in the Sign $+$ of the Quantity yy, from the other. Here the conſtant Part $\frac{a}{2}$ is the ſame as before, but the fluxionary Part $\frac{b\dot{y}}{\sqrt{bb + yy}}$ is different, or is not the Fluxion of an Arch

of

or Angle; and therefore its Fluent muſt be ſought in the Meaſure of a Ratio, by reducing it to one of the Forms in the Tables of Mr. *Cotes*. The Form which it agrees with, is the Sixth, *viz.* $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{\sqrt{e+fz^n}}$ to which the Fluxion $\frac{ab\dot{y}}{2\sqrt{bb+yy}}$ is reducible by the following Subſtitutions,

$$\textit{Viz.}\left\{\begin{array}{l} d=\frac{ab}{2} \\ z=y \\ r=o \\ n=2 \\ e=bb \\ f=1 \end{array}\right\}\begin{array}{c}\text{Alſo at the Bottom of the} \\ \text{Table you find}\end{array}\left\{\begin{array}{l} \mathrm{R}=\sqrt{f}=1. \\ \mathrm{T}=\sqrt{\frac{e+fz^n}{z^n}}=\frac{1}{y}\sqrt{bb+yy} \\ \mathrm{S}=\sqrt{\frac{e}{z^n}}=\frac{b}{y}. \\ \text{Alſo }\frac{2}{nf}d\mathrm{R}=\frac{ab}{2}. \end{array}\right.$$

Therefore in the Table the Fluent which ſtands againſt $r=o$, is $\frac{2}{nf}d\mathrm{R}\,\Big|\,\frac{\mathrm{R}+\mathrm{T}}{\mathrm{S}}=\frac{ab}{2}\,\Big|\,\frac{y+\sqrt{bb+yy}}{b}=\frac{ab}{2}$ L R. Now ſince b, and y are given, the Logarithm L of the Ratio of $y+\sqrt{bb+yy}$ to b, will be given alſo; whence the Fluent $\frac{ab}{2}$ L R may eaſily be computed for the Area of Sector A C D. (See Art. 882.)



892. The Reaſon why the Fluent of the Fluxions of the *Circular and Elliptic Sectors* were not found in the Meaſure of a Ratio, is evident, becauſe there f was a negative Quantity, or $f=-1$, and ſo it would have been $\mathrm{R}=\sqrt{-1}$, which is impoſſible, there being no ſuch Thing as the Square Root of a negative Quantity.

893. If the Tangent A T had been aſſumed for the variable Quantity, inſtead of the Sine P D; then the Fluxion of the *Circular Sector* would have been $\frac{aa a\dot{y}}{2aa+2yy}$; for the *Elliptic Sector*, $\frac{abb\dot{y}}{2bb+2yy}$; and for the Hyperbolical Sector

$\frac{abb\dot{y}}{2bb-2yy}$. The Form in Mr. *Cotes*'s Table for this, is the Second, *viz.* $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{e+fz^{n}}$;

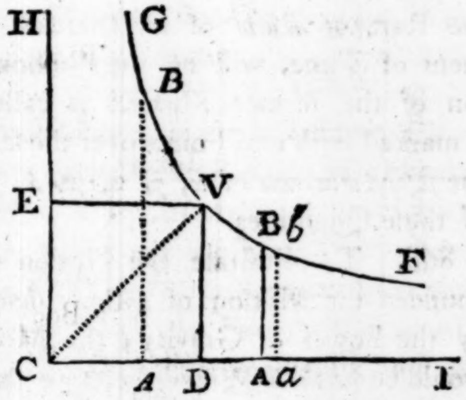
$$\text{Now putting}\left\{\begin{array}{l} d=abb\\ z=y\\ r=o\\ n=2\\ e=2bb\\ f=-2\end{array}\right\}\text{Then}\left\{\begin{array}{l} R=\sqrt{\frac{-e}{f}}=b.\\ T=z^{\frac{1}{2}n}=y.\\ S=\sqrt{\frac{e+fz^{n}}{f}}=\sqrt{bb-yy}\\ \text{And, }\frac{2}{ne}dR=\frac{1}{2}ab.\end{array}\right.$$

The Fluent againſt $r=o$ is $\frac{2}{ne}dR\left|\frac{R+T}{S}\right.=\frac{1}{2}ab\left|\frac{b+y}{\sqrt{bb-yy}}\right.=\frac{1}{2}abLR$; which will give the Area of the Sector A C D, in the Hyperbola, the ſame as before; but it will not be the Fluent or Area of the Circle or Ellipſis, becauſe there f is Affirmative, or $+f$, and therefore $R=\sqrt{\frac{-e}{f}}=-b$, which is impoſſible.

894. But ſince $\frac{aa\dot{y}}{aa+yy}$, or $\frac{bb\dot{y}}{bb+yy}$ are the Fluxions of an Arch, their Fluents will be akD, and bkD, and therefore $\frac{aa}{2}kD$ and $\frac{ab}{2}kD$ will be the Fluents or Areas of the Circular and Elliptic Sectors, the ſame as before.

895. That $\frac{b\dot{y}}{bb-yy}$ is the Fluxion of the Logarithm of $\frac{b+y}{\sqrt{bb-yy}}$ will be evident, if we conſider that the Fluxion of the Logarithm of $b+y$ is $\frac{\dot{y}}{b+y}=\frac{b\dot{y}-y\dot{y}}{bb-yy}$, and the Fluxion of the Logarithm of $\sqrt{bb-yy}=\frac{-y\dot{y}}{bb-yy}$. Now ſince the Fluxions of Logarithms have the ſame Properties with the Logarithms themſelves (861), if we ſubtract $\frac{-y\dot{y}}{bb-yy}$ from $\frac{b\dot{y}-y\dot{y}}{bb-yy}$, there will remain $\frac{b\dot{y}}{bb-yy}=$

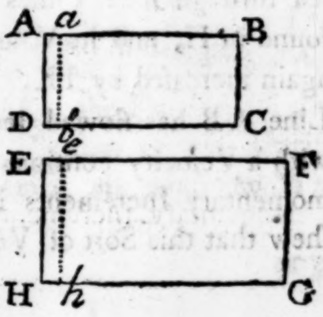
Fluxion of the Logarithm of $\frac{b+y}{\sqrt{bb-yy}}$, which was to be ſhewn.



896. It will be worth while to ſhew how eaſily the Fluent, or aſymptotic Space ADVB is obtained by this Method from the Fluxion $\dot{x}\,y = \frac{aa\dot{x}}{a+x}$, (See Art. 829), which comes under Mr. *Cotes*'s *firſt Form, viz.* $\frac{d\dot{z}z^{rn-1}}{e+fz^n}$, for putting $d=aa$, $z=x$, $n=1$, $r=1$, $e=a$, $f=1$; we have (againſt $r=1$,) the Fluent $\frac{d}{nf}\left|\frac{e+fz^n}{e}\right. = aa\left|\frac{a+x}{a}\right.$ $= CD^2\left|\frac{CD+AD}{CD}\right. = 1\left|\frac{1+x}{1}\right. = LR = 2.302585L$ = Area of the Space ADVB, which therefore is given from the Logarithm of the Number $1+x$ in the common Tables. Hence it is again evident, that the hyperbolic Space ADVB is the Logarithm of $1+x$.

CHAP. XVII.

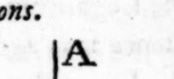
The Nature of SECOND, THIRD, *&c.* FLUXIONS *explained.*



897. FROM what we have ſaid of the Nature and Notation of Fluxions in general, it is evident, when the Velocity is always the ſame, the fluent Quantities AB and EF can have but one Sort of Fluxions A *a* and E *e*, which will be to each other, as the Velo-

cities of the Fluents reſpectively. But if the Velocity with which the Fluents A B and E F increaſe or decreaſe be not conſtant, but variable every Moment, then may this Velocity itſelf be conſidered as a *Fluent*, or *flowing Quantity*; and the Rate or *Ratio* of its Increaſe or Decreaſe each Moment of Time, will be the Fluxion thereof. And this Fluxion of the former Fluxion is called a *ſecond* Fluxion, and is marked with two Points over the ſame Letter; thus $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are the *ſecond Fluxions* of x, y, z, as $\dot{x}$, $\dot{y}$, $\dot{z}$, are the *firſt Fluxions* of thoſe Quantities.

898. To illuſtrate the Notion of *ſecond Fluxions*, let us conſider the Motion of a Body deſcending by its Weight, or by the Power of Gravity; the Motion reſulting from hence would be *accelerated*, and therefore the Velocity continually increaſing, (ſuppoſing the Fall of the Body to be in *Vacuo*) conſequently the Velocity of the Body the firſt Moment, (which is the *firſt Fluxion*) would be augmented the ſecond Moment, by the continual Action of Gravity, and this Augmentation will be the *ſecond Fluxion*, or *Fluxion of the firſt Fluxion*; and ſince this Acceleration or Increaſe of Velocity is regular and uniform, or proportional to the Times, 'tis plain this Sort of Motion will admit of nothing beyond *ſecond Fluxions*.

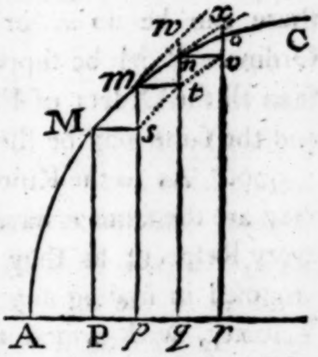


899. *For Example*; ſuppoſe from the loweſt Point B of the Line A B, an heavy Body begins to deſcend, and at the End of the firſt Moment it arrives to C, let the Velocity it then has, be expreſſed by the Line C D. At the End of the ſecond Moment it will have paſſed through four Times that Space, and be found in E, its Velocity then will be E G, increaſed by F G. At the End of the third Moment, it will have paſſed through nine Times the firſt Space, and be found in H, and its Velocity will then be H K, again increaſed by I K. In all this Time the Line A B has flowed into the Line A H, but with a Velocity continually increaſing; and its momentary Increments F G, I K being equal, ſhew that this Sort of Velocity admits of no further Variation,

and therefore of *Second Fluxions* only, or rather of *Fluxion*, in the abſolute Senſe; for I ſuppoſe I need not tell the Reader, that $\ddot{x}$ is but the ſame Fluxion of $\dot{x}$, as $\dot{x}$ is of x; and that $\ddot{x}$ has no Relation to the Fluxion of the Line A B, or x, but to the Velocity C D, or $\dot{x}$. The Want of this Hint in other Writers has left the young Student very often in the utmoſt Ambiguity and Confuſion.

900. The Fluxions of ſecond Fluxions are called *third Fluxions*; and are thus marked with three Points $\dddot{x}$, $\dddot{y}$, $\dddot{z}$. Theſe take Place in the Motion of an heavy Body deſcending by its Gravity at a great Diſtance above the Surface of the Earth; for in ſuch a Caſe, its Weight, or Power of Gravity increaſes continually, as the Squares of the Diſtances from the Earth's Center decreaſes: And therefore the momentary Increments F G of the firſt Velocity, (which before in ſmall Diſtances, near the Surface of the Earth, were eſteemed equal) will now receive a continual Increaſe, or Acceleration themſelves; and hence *third Fluxions* will ariſe. And ſince the nearer the Body approaches the Earth, the greater will be its Weight, theſe Accelerations will be conſtantly increaſed; and hence every Moment will produce a new Order of Fluxions, or *Third*, *Fourth*, *Fifth*, &c. *Fluxions*, will be generated *in Infinitum*, or ſo long as the Motion continues, as is eaſy to conceive.

901. What has been already ſaid, in regard to *Second* Fluxions, may be farther illuſtrated and explained, by conſidering the Ordinate P M, as moving along the Axis A B, of the Curve A M C, from A towards B: Let A P $= x$, P M $= y$, and A M $= z$.



902. If now in an indefinite ſmall Particle of Time, the Ordinate P M moves from that Situation to another $p\,m$; then, drawing M s parallel to A B, we ſhall eaſily obſerve, that $s\,m$ will be the Velocity, or Fluxion of P M, and M m the Fluxion of the Arch A M, and alſo that M s is the Fluxion of the Abſciſs A P, in the ſame Time; that is, M $s = \dot{x}$, $m\,s = \dot{y}$, and M m $= \dot{z}$.

903.

903. If in the next Moment the Ordinate move to $q\,n$; draw $m\,t$ parallel to AB, join $s\,t$, and draw parallel thereto $m\,w$, to interſect the Ordinate $q\,n$ in w; then will the Velocity with which the ſaid Ordinate decreaſeth in the ſecond Moment be $n\,t$, and the Difference between $m\,s$ and $n\,t$, or $w\,t$ and $n\,t$, *viz.* $w\,n$, will be the Variation of the Velocity, or Fluxion $m\,s$, that is, it will be the *Second Fluxion* of PM; and ſo $w\,n = \ddot{y}$.

904. Again, in the third Moment, ſuppoſe the Ordinate arrives to the Situation $r\,o$, draw $n\,v$ parallel to AB; join $t\,v$, to which draw $n\,x$ parallel, meeting the Ordinate $r\,o$ in x; then will $v\,o$ be its Fluxion the third Moment, which is leſs than that in the Second, by the Quantity $x\,o$, (becauſe $n\,t = x\,v$) therefore the Difference between $w\,n$ and $x\,o$, will be the Variation of the Velocity $w\,n$ or $\ddot{y}$ in this laſt Moment; and ſo $w\,n - x\,o = \dddot{y}$, the Fluxion of $\ddot{y} = w\,n$, or *third Fluxion* of PM. And after this Manner you may eaſily conceive, how *Fourth*, *Fifth*, &c. *Fluxions are generated.*

905. If the Spaces $P\,p$, $p\,q$, $q\,r$, are ſuppoſed equal, and paſſed over in equal Times or Moments, then will the Abſciſs AP flow uniformly, and its momentary Velocities or Fluxions $M\,s$, $m\,t$, $n\,v$, *&c.* will be all equal; and conſequently $\dot{x}$ will, in this Caſe, be a conſtant or invariable Quantity; and ſo there can be no $\ddot{x}$, or ſecond Fluxion of AP. But if the Ordinate PM be ſuppoſed to flow with a variable Velocity, then all the Orders of Fluxions will ariſe, as we have ſhewn; and the ſame may be ſhewn of the Arch AM.

906. As to the Rules for operating Second, *&c.* Fluxions, they are the ſame as have been laid down for the *firſt Fluxions* in every Reſpect; as they needs muſt be, ſince nothing more is propoſed in finding any Fluxion, than diſcovering the Ratio of Velocity, with which the Fluent varies its Magnitude, in becoming greater or leſſer: And as this is the ſame Thing in every Kind of Fluent, ſo the Rules for determining it muſt in every Order of Fluxions be the ſame.

907. Hence as the Fluxion of x is $\dot{x}$, ſo that of $\dot{x}$ is $\ddot{x}$; and that of $\ddot{x}$ is $\dddot{x}$; and ſo on.

Note here, that when any of theſe firſt Fluxions are ſuppoſed to be conſtant, they will be expreſſed in *Roman* Characters, as $\dot{\mathrm{x}}$, $\dot{\mathrm{y}}$, $\dot{\mathrm{z}}$; but if they are flowing Quantities, they will be repreſented as uſual in *Italics*, as $\dot{x}$, $\dot{y}$, $\dot{z}$.

908. Thus the Fluxion of $\dot{\mathrm{x}}\dot{y}$ is $\dot{\mathrm{x}}\ddot{y}$; and of $\dot{x}\dot{y}$, is $\dot{x}\ddot{y} + \dot{y}\ddot{x}$, and the Fluxion of $\dot{x}\dot{y}\dot{z}$ is $\dot{x}\dot{y}\ddot{z} + \dot{x}\dot{z}\ddot{y} + \dot{y}\dot{z}\ddot{x}$. Alſo the Fluxion $x\dot{y}$ is $x\ddot{y} + \dot{y}\dot{x}$, and therefore the Fluxion of $x\dot{y} + y\dot{x}$, is $x\ddot{y} + \dot{y}\dot{x} + y\ddot{x} + \dot{y}\dot{x}$, or $x\ddot{y} + y\ddot{x} + 2\dot{x}\dot{y}$; which, as it is the firſt Fluxion of $x\dot{y} + y\dot{x}$, (as that is of xy,) ſo it will be the ſecond Fluxion of xy. And thus the ſecond Fluxion of any other Quantity may be found, as *per* Rule (793).

909. The ſecond Fluxion of any *Power* of any primary Fluxion, is found by the Rule for Powers (794). Thus the Fluxion of $\dot{x}^2$ is $2\dot{x}\ddot{x}$; and of $\dot{x}^3$, is $3\dot{x}^2\ddot{x}$. Alſo the Fluxion of $2x\dot{x}$ (the Fluxion of x^2) is $2x\ddot{x} + 2\dot{x}\dot{x}$; which therefore is the ſecond Fluxion of x^2.

910. The Fluxions of *fluxionary Fractions* are alſo found by the proper Rule (801). Thus the Fluxion of $\frac{\dot{x}}{\dot{y}}$ will be found to be $\frac{\ddot{x}\dot{y} - \dot{x}\ddot{y}}{\dot{y}^2}$; the Fluxion of $\frac{\dot{\mathrm{x}}}{\dot{z}}$, is $\frac{-\dot{\mathrm{x}}\ddot{z}}{\dot{z}^2}$; of $\frac{1}{\dot{x}}$, is $\frac{-\ddot{x}}{\dot{x}^2}$; and of $\frac{y\dot{y}}{\dot{x}}$, is $\frac{\dot{x}y\ddot{y} + x\dot{y}^2 - y\dot{y}\ddot{x}}{\dot{x}^2}$; but if it be $\frac{\mathrm{y}\dot{y}}{\dot{x}}$, the Fluxion is $\frac{\mathrm{y}\ddot{y} - \mathrm{y}\dot{y}\ddot{x}}{\dot{x}^2}$; if $\frac{y\dot{\mathrm{y}}}{\dot{x}}$, it is $\frac{\dot{y}\dot{\mathrm{y}} - y\mathrm{y}\ddot{x}}{\dot{x}^2}$; if $\frac{y\dot{y}}{\dot{\mathrm{x}}}$, the Fluxion is $\frac{\dot{\mathrm{x}}y\ddot{y} + \dot{\mathrm{x}}\dot{y}\dot{y}}{\dot{\mathrm{x}}^2} = \frac{y\ddot{y} + \dot{y}^2}{\dot{\mathrm{x}}}$; and ſo of others.

911. The *ſecond Fluxion* of a *ſurd Quantity* is found in the ſame Manner as the Firſt (798); thus the Fluxion of $\sqrt[2]{\dot{x}} = \dot{x}^{\frac{1}{2}}$, is $\frac{1}{2}\dot{x}^{-\frac{1}{2}}\ddot{x} = \frac{\ddot{x}}{2\dot{x}^{\frac{1}{2}}}$; of $\sqrt[3]{\dot{x}^2} = \dot{x}^{\frac{2}{3}}$, is $\frac{2\ddot{x}}{3\dot{x}^{\frac{1}{3}}}$; the Fluxion of $\frac{\dot{x}}{2x^{\frac{1}{2}}}$ (which is itſelf the Fluxion of $\sqrt[2]{x}$) is $\frac{2\ddot{x}x^{\frac{1}{2}} - x^{-\frac{1}{2}}\dot{x}^2}{4x}$, which therefore is the *ſecond Fluxion* of $\sqrt[2]{x}$, and which, if we divide by $4x$, will be thus expreſſed, $\frac{1}{2}x^{-\frac{1}{2}}\ddot{x} - \frac{1}{4}x^{-\frac{2}{3}}\dot{x}^2$.

912. We have ſhewn that the Fluxion of the Logarithm of

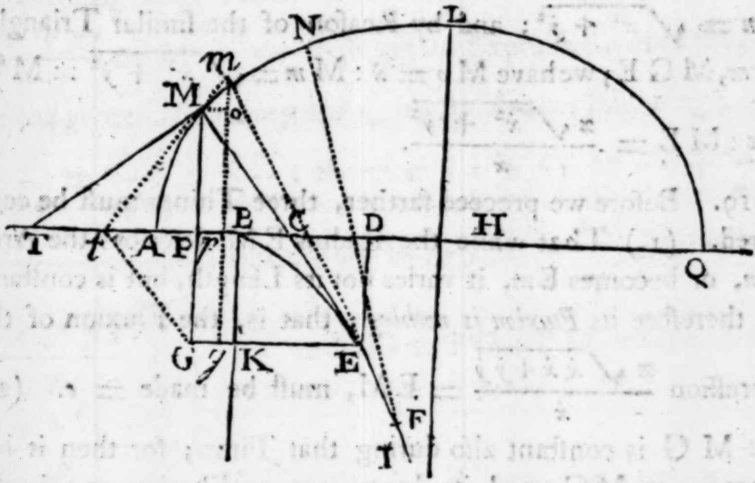
$1 + x$ is $\frac{\dot{x}}{1 + x}$, and the Fluxion of this is $\frac{\ddot{x} + \ddot{x}x - \dot{x}\dot{x}}{1 + 2x + xx}$, which is therefore the ſecond Fluxion of the Logarithm of that Quantity: And thus I preſume the *Second*, *Third*, &c. *Fluxions* of any Quantity propoſed, may be eaſily found by any one who underſtands the Rules for the *firſt Fluxions*, without any further Directions.

CHAP. XVIII.

Of INVOLUTE *and* EVOLUTE Curves; *and the Method of inveſtigating the* Radii *of* EVOLUTION, *or* CURVATURE.

913. BY the Problems of this Chapter, it will appear how great and extenſive the Uſe of the Doctrine of Fluxions is, both of the *Firſt and Second Orders*, ſince their Solutions are hereby rendered facile and perſpicuous, which by the *common Geometry* were not only difficult and tedious, but, to an ordinary Genius, quite inacceſſible, or inſuperable. Theſe relate to the Nature of *Involute and Evolute Curves*, and the Methods of finding the *Radii of Evolution, or Curvature.*

914. If in the Point A, in the Right Line A H, we ſuppoſe one End of a flexible String, or Thread to be faſtened; and Part thereof to lie along, or be coincident with the ſaid Line from A to B, and the other Part to be ſtretched along the Curve B E F, and there fixed at I. And then if the other End A be looſened, and raiſed from its Poſition on the Line and Curve, and moved tight and ſtrait along, while it diſengages itſelf from the Curve B E I, then the Point A will deſcribe a Curve A M N, which is called an *Involute Curve.* And the Curve B E I, is called the *Evolute* of the Curve A M N, or the *Evolute Curve.* And the ſtrait Lines E M, F N, are called the *Radii of Evolution, or Curvature* for the Points M and N.



915. It is evident (ſince the Length of the Thread A B E I continues the ſame) that the Part of the Evolute E F is equal to the Difference of the *Radii* F N and E M or that E F = F N — E M. Thus alſo B E = E M — A B. And when A B = *o*, the Parts of the Curve B E, B F will be equal to their reſpective *Radii* E M and F N.

916. 'Tis moreover evident, that each Point of the Curve A M N is deſcribed by a *different Radius*, which is perpendicular to it in that Point. Thus the Points A, M, N, are deſcribed with the different *Radii*, A B, E M, F N, on the reſpective Centers B, E, F, and conſequently the Curvature in thoſe Points will be reciprocally as the Length of thoſe *Radii*.

917. Therefore the *Radii* belonging to each Point of the *Involute* A M N, by their various Interſections form the Curve of the *Evolute* B E F. Thus, ſuppoſe the two Points M and *m* contiguous, their *Radii* touch each other in the Point E, and if continued out, would there interſect each other. Therefore,

918. It is a *general Problem; The Nature of the Curve* A M N *being given, to find the Length of the Radius of Evolution* E M, *belonging to a given Point* M.

Let the Ordinate P M, *p m*, be continued out, and from the Point E of the Interſection of the *Radii*, to the Points M and *m*, draw E G perpendicular to the Ordinates, and draw E D parallel to P G; and *m o* to A B. Let A P = *x*, P M = *y*, and M G = *z*, and we have M *o* = $\dot{x}$, *m o* = $\dot{y}$, and

$Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$; and by Reaſon of the ſimilar Triangles Mom, MGE; we have $Mo = \dot{x} : Mm = \sqrt{\dot{x}^2 + \dot{y}^2} :: MG = z : ME = \frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$.

919. Before we proceed farther, three Things muſt be conſidered. (1.) That while the Radius EM deſcribes the Arch Mm, or becomes Em, it varies not its Length, but is conſtant, and therefore its *Fluxion is nothing*; that is, the Fluxion of the Expreſſion $\frac{z\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}} = EM$, muſt be made $= o$. (2.) $z = MG$ is conſtant alſo during that Time; for then it becomes $og = MG$, and its Increment or Fluxion mo is the Fluxion of PM, that is, $\dot{z} = \dot{y}$. (3.) Since the Fluxion of the Abſciſs reſults from the Motion of the Thread, which we ſuppoſe to be equable and uniform, 'tis plain it will be conſtant, that is $Mo = \dot{x}$, will be the ſame in all equal Spaces of Time.

920. Therefore making $\dot{x}$ conſtant, and the Fluxion of $EM = \frac{z\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$, *viz.* $\dot{z} \times \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} + \frac{z\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} = \frac{\dot{z}\dot{x}\dot{x} + z\dot{y}\dot{y} + z\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} = o$, which dividing by $\dot{x}$, and putting $\dot{z} = \dot{y}$, (919.) we have $\frac{\dot{y}\dot{x}^2 + \dot{y}^3 + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}}} = o$, that is, $\dot{y}\dot{x}^2 + \dot{y}^3 + z\dot{y}\ddot{y} = o$, and ſo $\dot{x}^2 + \dot{y}\dot{y} = -z\ddot{y}$, and thus we get $z = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}} = MG$. Hence a Perpendicular to the Point G, will interſect the Tangent MC in E, which will give ME, the Radius of Evolution required.

921. *To find the Radius of Evolution for any Point* M, *in the Curve of a Parabola* AMN.

The Equation of the Curve is $px = y^2$, which in Fluxions gives $\dot{y} = \frac{p\dot{x}}{2y} = \frac{p\dot{x}}{2\sqrt{px}}$, and ſo $\ddot{y} = \frac{-2p\dot{x}\dot{y}}{4yy} = \frac{-2p\dot{x}\dot{y}}{4px} = \frac{-2p\dot{x}}{4px} \times \dot{y} = \frac{-2p\dot{x}}{4px} \times \frac{p\dot{x}}{2\sqrt{px}} = \frac{-p\dot{x}^2}{4x\sqrt{px}}$. Now ſubſtituting theſe Values of $\dot{y}$ and $\ddot{y}$, in the general Expreſſion $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{-\ddot{y}} = \text{M G}$, we ſhall get an Expreſſion, conſiſting only of the given Quantities p and x, in the Manner following; $\dot{y}\dot{y} = \frac{p^2\dot{x}^2}{4yy} = \frac{p^2\dot{x}^2}{4px} = \frac{p\dot{x}^2}{4x}$; therefore $\dot{x}\dot{x} + \dot{y}\dot{y} = \dot{x}\dot{x} + \frac{p\dot{x}\dot{x}}{4x} = \frac{4\dot{x}^2x + 4xp\dot{x}}{4x}$, this divided by $-\ddot{y} = \frac{p\dot{x}^2}{4x\sqrt{px}}$, will give $\frac{4x\sqrt{px}}{p} + \sqrt{px} = \text{M G} = \text{P G} + \text{P M}$, whence $\text{P G} = \frac{4x\sqrt{px}}{p} = \text{D E}$.

922. Let t M touch the Parabola in M, then is $\text{t P} = 2x$, and $\text{P C} = \frac{1}{2}p$; for $\text{t P}\ (2x) : \text{P M}\ (y) :: \text{P M}\ (y) : \text{P C} = \frac{yy}{2x}$, therefore $2x \times \text{P C} = yy = px$, conſequently $\text{P C} = \frac{1}{2}p$. And as $\text{P M}\ (y) : \text{P C}\ (p\tfrac{1}{2}) :: \text{M G}\left(\frac{4x\sqrt{px}}{p} + \sqrt{px}\right) : \text{G E} = \text{P D} = \frac{1}{2}p + 2x = \text{P C} + \text{C D}$; but $\text{P C} = \frac{1}{2}p$, therefore $\text{C D} = 2x = \text{P t}$.

923. Now ſince $\text{P C} = \frac{1}{2}p$, is an invariable Quantity, therefore when the Point M coincides with the Vertex A, the Point P will be there too, and conſequently P C will then become $\text{A B} = \frac{1}{2}p$. *Hence the Vertex* B *of the Evolute* B E F, *touches the Axis of the Parabola, at the Diſtance of* $\text{A B} = \frac{1}{2}p$, *from the Vertex* A.

924. Hence the *Radius of Evolution* in the *Parabola*, when a *Minimum*, is $= \text{A B} = \frac{1}{2}p =$ Half the Parameter, conſequently the *Curvature of the Parabola* is then and there a *Maximum*, and *equal to the Curvature of a Circle deſcribed with the Radius* $\text{A B} = \frac{1}{2}p$.

925. To find the Curve (or Nature) of the Evolute B E I.

We have already ſeen that $\text{B K} = \text{P G} = \frac{4x\sqrt{px}}{p}$; and $\text{K E} = \text{B D} = \text{A P} + \text{P D} - \text{A B} = x + \frac{1}{2}p + 2x - \frac{1}{2}p = 3x$. Let the Abſciſs $\text{B K} = x$, and the Ordinate $\text{K E} = y$; then

$\dot{x}$;

$x : y :: \frac{4x\sqrt{px}}{p} : 3x$, whence $3xx = \frac{4xy\sqrt{px}}{p}$. But $y = 3x$, and ſo $x = \frac{y}{3}$, which ſubſtituted for x in the Equation above, there comes out $\frac{27}{16}pxx = y^3$, which ſhews the Evolute B E I is a *ſecond cubical Parabola*, whoſe Parameter is $\frac{27}{16}$ of the Parameter of the Involute Parabola A M N.

926. Hence it appears, that the *Evolute* is a *Geometrical Curve* when the *Involute* is ſuch; or an Equation may be found to expreſs the Relation of its Abſciſſes B K, to its Ordinates K E in finite Terms, and free from Fluxions, when the Involute has ſuch an Equation.

927. Having found the *Radius of Evolution* M E, we rectify the Part of the Evolute B E with eaſe. For it will always be, the Curve B E = M E — A B, *viz. the Curve* B E *will always be equal to the Difference of two given right Lines.*

928. We ſhould next have proceeded to the *Ellipſis* and *Hyperbola*, but that in our Method we muſt have Recourſe to the Radius of Evolution directly M E, which becauſe of the ſimilar Triangles M *o m*, M G E, will be found = $\frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$; which will prove too troubleſome; and therefore, as well as for Eaſe as Variety, we will inveſtigate another Expreſſion for the Radius M E, in a different Manner than before.

929. In order to this, let M G = v, and the Arch A M = $\dot{z}$; the Reſt as before. Then $\dot{x} : \dot{z} :: v : \frac{v\dot{z}}{\dot{x}}$ = M E, which as it is conſtant, its Fluxion is nothing, therefore the Fluxion of $\frac{v\dot{z}}{\dot{x}}$ is $\frac{\dot{v}\dot{z}\dot{x} + \ddot{z}v\dot{x} - v\dot{z}\ddot{x}}{\dot{x}^2} = 0$. Therefore $\dot{v}\dot{z}\dot{x} = v\dot{z}\ddot{x} - v\dot{x}\ddot{z}$; and ſo $v = \frac{\dot{v}\dot{z}\dot{x}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = \frac{\dot{z}\dot{x}\dot{y}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}}$, becauſe $\dot{v} = \dot{y}$.

930.

930. Therefore $\dot{x} : \dot{z} :: \frac{\dot{x}\dot{y}\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} : \frac{\dot{y}\dot{z}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = \text{ME}$; and $\dot{x} : \dot{y} :: \frac{\dot{x}\dot{y}\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} : \frac{\dot{y}^2\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = \text{GE} = \text{P}D$, and ſo $\frac{\dot{z}\dot{y}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} + x = \text{PD} + \text{AP} = \text{AD}$.

Laſtly, $\frac{\dot{x}\dot{y}\dot{z}}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} - y = \text{MG} - \text{AP} = \text{PG} = \text{BK}$.

In all theſe Expreſſions, when either $\dot{x}$, $\dot{y}$, or $\dot{z}$, are conſtant, they may have Unity placed in their Stead, and the Terms when in $\ddot{x}$, $\ddot{z}$, $\ddot{y}$, are found, will vaniſh, and ſo the above Expreſſions will be abridged very conſiderably. All which will be plain by the following Examples.

931. Let AM be an Arch of the Curve of an *Ellipſis*, or *Hyperbola, to find the Radius of Evolution* EM. The Equation for theſe Curves being $\frac{aa}{bb}yy = ax \mp xx$, (where a and b are the tranſverſe and conjugate Diameters,) we have $y = \frac{b}{a}\sqrt{ax \mp xx}$; and $\dot{y} = \frac{ab \mp 2bx}{2a\sqrt{ax \mp x^2}}$; and $-\ddot{y} = \frac{ba}{4 \times \overline{ax \mp xx}^{\frac{3}{2}}}$. Now by ſubſtituting theſe Values of y, $\dot{y}$, $\ddot{y}$, into the general Equations above, we ſhall have the Radius of Evolution, and the other Parts expreſſed by them.

332. Thus $\frac{\dot{y}\dot{z}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} = \text{ME}$, by making $\dot{x} = 1$, or $\ddot{x} = 0$, will become $\frac{\dot{y}\dot{z}^2}{-\ddot{z}}$; and ſince $\dot{z} : \dot{y} :: \dot{y} : \ddot{z}$* $= \frac{\dot{y}\ddot{y}}{\dot{z}}$, therefore $\frac{\dot{y}\dot{z}^2}{-\ddot{z}} = \frac{\dot{z}^3}{-\ddot{y}}$. But $\frac{\dot{z}^3}{-\ddot{y}} = \frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$, in which Equation the Values of $\dot{y}$, and $-\ddot{y}$, above found, being ſubſtituted will give the Value of the Radius of Evolution ME.

933. Again, $\frac{\dot{z}\dot{y}^2}{\dot{z}\ddot{x} - \dot{x}\ddot{z}} + x = \text{AD}$, becomes $\frac{\dot{z}\dot{y}^2}{-\ddot{z}} + x = \frac{\dot{z}^2\dot{y}}{-\ddot{y}}$

---

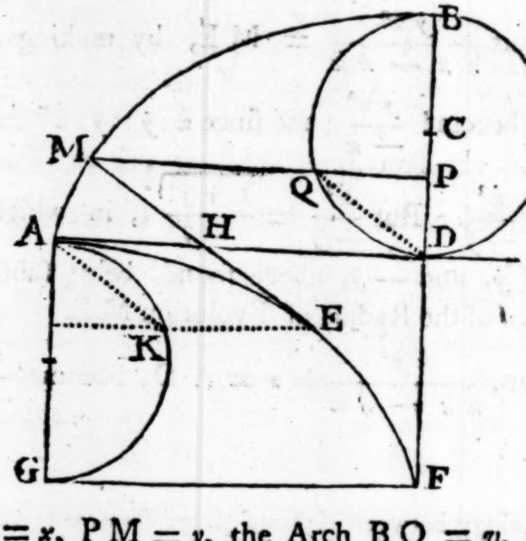
* This Analogy between *firſt and ſecond Fluxions* is eaſily deduced from the Figure to Art. 901, where, if on the Center *m*, a ſmall Arch be deſcribed from *n* to *m w*, it will conſtitute a *ſecond fluxionary Triangle*, ſimilar to the firſt *m t w*, which give the Analogy above.

$\frac{\dot{z}^2\,\dot{y}}{-\ddot{y}} + x = \mp\frac{a}{2} + \frac{\overline{b^2 \mp a^2 + \overline{a \mp 2x}^{\,3}}}{2a^4} =$ A D; this, if we put $x = 0$, will become $\frac{bb}{2a} =$ A B, that is, when the Point M coincides with the Vertex A, the Radius of Evolution then becomes a *Minimum*, and equal to A B = half the Parameter of the Axis A Q.

934. If this Quantity $\frac{bb}{2a}$ be taken from the above Expreſſion for A D, it will leave $\frac{\mp a^2 - b^2}{2} + \frac{\overline{b^2 \mp a^2 \times \overline{a \mp 2x}^{\,3}}}{2a^4}$ = K E = B D; if here $b = a$, the Ellipſis will become a Circle, and conſequently the Points E and D will ever coincide with the Point B; and therefore the Evolute B E of a Circle will degenerate into a Point, *viz.* the *Center* thereof.

935. If in the Expreſſion gotten for M E, as above directed, we make $x = \frac{1}{2}a =$ A H, that Expreſſion will become $= \frac{aa}{2b}$ = half the Parameter of the conjugate Axis, and the Radius of Evolution, is then a *Maximum*, and the Curvature of the Ellipſis a *Minimum*.

936. *To find the Evolute* A E F, and *Radius of Evolution* M E of the *common Cycloid* A M B.



Let B P = x, P M = y, the Arch B Q = v, and B D = $2a$; then P Q = $\sqrt{2ax - xx}$, by the Property of the Circle;

and $PM = y = v + \sqrt{2ax - xx}$, by the Property of the Cycloid *. This laſt Equation in Fluxions will be $\dot{y} = \dot{v} + \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$; but $\dot{v} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$, (875.) therefore $\dot{y} = \frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}} = \dot{x}\frac{\sqrt{2a - x}}{x}$, and $\ddot{y} = \frac{-a\dot{x}^2}{x\sqrt{2ax - xx}}$, by making $\dot{x}$ invariable. Therefore ſubſtituting theſe Values of $\dot{y}$ and $\ddot{y}$ in the general Expreſſion for $ME = \frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}} = \frac{\overline{1 - \dot{y}^2}\sqrt{1 - \dot{y}^2}}{-\ddot{y}}$, we ſhall get $ME = 2\sqrt{4aa - 2ax} = 2DQ = 2MH$.

937. If we put $x = o$, we have $ME = 2\sqrt{4aa} = 4a$ when the Point M coincides with the Vertical Point B. But if $x = 2a$, then $ME = o$, when the Point M coincides with A. Therefore the Point A is the Beginning of the Evolute, and the Point F the End, and ſince $FB = 4a$, FD is $= 2a = DB$. Compleat the Rectangle DG; deſcribe the Semicircle AKG; and draw AK parallel to ME or DQ. Then ſince the Angle $DAK = ADQ$, the Arches $AK = DQ$, and the Chord $AK = DQ$; whence KE is parallel and equal to AH, and therefore equal to the Arch DQ, as is evident from the Generation of the Cycloid; therefore, alſo EK is equal to the Arch AK, which is the noted Property of the Cycloid. *Therefore the Evolute* AEF *is a Semi-cycloid every alike and equal to the Involute* AMB, *but in an inverted Poſition.*

Thus I have at Length finiſhed what I have thought neceſſary to premiſe, as an Introduction to the mathematical Sciences, the fundamental Principles of Arithmetic, Logarithms, Algebra, Geometry, Conic Sections and Fluxions, in nearly one Thouſand Articles. This Syſtem of mathematical Elements will, if impartially conſidered, be found a ſhorter and more eaſy Introduction to the Practice of thoſe Arts which depend on them, than any other that has been hitherto publiſhed,

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* This we ſhall demonſtrate, when we give an Account of the *Geneſis* and *Properties* of *mechanical*, or *tranſcendent Curves*, in our Introduction to *Mechanical Philoſophy*.

liſhed, that I know of. The Tranſition from one Branch to the other is natural, and ſhews their mutual Connection and Dependancy, and the Reaſon of the ſubſequent Articles evidently appears from the preceding ones: Nor do I know, that I have any where advanced one ſingle Poſition, that has not been previouſly demonſtrated. Such a Series of preliminary Principles appears to me abſolutely neceſſary, to qualify every Learner for a rational underſtanding of the moſt uſeful Sciencies that are now to follow, and which we can now communicate in a more compendious, and leſs expenſive Form, and in a pleaſanter Manner, than we could have done any other Way. If theſe Inſtitutions are thought to be dry or uſeleſs, it muſt be owing to a Want of Genius, or a natural Incapacity or Diſinclination to ſuch Studies. We do not here treat of Trifles, and what we write is intended, in this Part, for ſuch only, who are *born Mathematicians.* They will have a natural Reliſh for, and take a peculiar Delight in them. It might have been expected, that a *Table* of *Logarithms* ſhould have been added, but we have judged that unneceſſary, as they are almoſt in every one's Hands, and as we could make no Alteration in that Subject, nor afford them cheaper than they may be bought by themſelves; it would be ſcarce fair, to oblige our Readers to buy the ſame Things over again; eſpecially, as the *artificial Line of Numbers, Sines and Tangents,* ſupply their Place, in moſt practical Caſes, by Inſtruments, which will now very ſoon follow in their proper Order and Places, among the practical mathematical Sciences, which will be the Subjects of the following Volume, where the Reader will find them treated of in a Method new, and with many Alterations and Improvements, both in the Subject Matter, and the Inſtruments for Practice.

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